Supplemental Material for *A Student's Guide to Laplace Transforms*

Convolution

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Introduction

The process of convolution is introduced in Section 3.8 in Chapter 3 of *A Student's Guide to Laplace Transforms* because of the Laplace-transform property relating convolution of two functions in the time domain to multiplication of the Laplace transforms of those two functions in the generalized-frequency (s) domain. That's quite useful, but the process of convolution is also important in a wide range of applications in engineering, physics, and applied mathematics. Those applications include signal processing, probability theory, image processing, differential equations, and many others.

If you're encountered convolution before, in may have been in the context of designing and analyzing linear time-invariant (LTI) systems. That's because the output of any LTI system is the convolution of the input signal and the impulse response of the system (that is, the response of the system to an input consisting of a single impulse – a time-domain delta function). So understanding the process of convolution is especially important for such systems.

The notation for convolution used in some texts (including of *A Student's Guide to Laplace Transforms*) looks a bit unusual at first: the convolution of two time-domain functions $f(t)$ and $g(t)$ is written as $(f * g)(t)$. In this expression, the asterisk signifies convolution, so the expression in the first set of parentheses could be written as $f(t) * g(t))$, but the result of the convolution process is also a function of time, and that's the meaning of the *(t)* portion of the expression $(f * g)(t)$. Hence the convolution could be written as $[f(t) * g(t)](t)$, for which $(f * g)(t)$ is a convenient shorthand.

Several useful properties of the convolution process can be conveniently written using this notation. Three of those properties are that the convolution process is

1) Commutative, so $(f * g)(t) = (g * f)(t)$
2) Associative, so $[f * (g * h)(t)] = [(f * g) * h)](t)$, and
3) Distributive, so $(f * g)(t) + (f * h)(t) = [f * (g + h)](t)$.

Another useful property is that the convolution of any function with the Dirac delta function simply returns the original function, so $(f * \delta)(t) = f(t)$.

The reasons that the convolution process has these properties can be understood by considering the mathematical equation that describes convolution. That's the subject of the next section of this document.

Section 2: Mathematical statements of convolution process

In Chapter 3 of the text, the mathematical statement of the convolution process for two time-domain functions is given as

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau \qquad \text{(Eq. 3.24 in text)}$$

in which both *f(t)* and *g(t)* are causal functions, so both have zero value for $t<0$.

As explained in the text, the lower limit of the integration over the time variable τ in Eq. 3.24 is determined by the fact that the causal function *f(τ)* is zero for τ<0 and the causal function *g(t-τ)* is zero for τ>t (since its argument is negative in that case).

For general (not necessarily causal) functions *f(t)* and *g(t)*, the equation for the convolution process is

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau. \quad (1)$$

in which the integration over τ extends over all time, from negative infinity to positive infinity.

As described in the text, the integral in Eq. 1 can be understood as the following series of steps:

1) Both *f(t)* and *g(t)* are written as functions of the dummy[1] time variable *τ*
2) One of the two functions, *g(τ)* in this case, is reversed in time, making *g(-τ)*
3) The function *g(τ)* is then offset in time by amount *t*, making *g(t-τ)*
4) The function *g(t-τ)* is multiplied by the other function (which is *f(τ)* in this case)
5) The multiplication products are added up (that's the integral over *τ*)
6) As time passes and *t* incrementally increases, the function *g(t-τ)* ``slides past" the function *f(τ)*.
7) At each time *t*, the point-by-point multiplication of *g(t-τ)* and *f(τ)* is repeated
8) The results of the multiplication are accumulated for each value of *t*.

These steps are easy to visualize, especially with the help of the graphical analysis shown in the next two sections of this document. Most students understand the convolution process as a mathematical procedure in which one of the functions is used to modify the other (since the functions are multiplied together at each value of time), but it's quite common for students to ask the question "What good does it do to reverse one of the functions before offsetting it and multiplying by the other function?"

One answer to that question is that it's perfectly possible to perform a similar operation in which neither function is reversed; this process is called the "cross-correlation" of the two functions, and cross-correlation is useful in providing a mathematical measure of the similarity of two functions. But by reversing one of the functions, the convolution process takes on several characteristics that make it useful in a variety of applications. The commutative property mentioned above is one of those characteristics, because reversing one of the functions ensures that $(f * g)(t) = (g * f)(t)$. You can verify that by writing

$$(g * f)(t) = \int_{\tau=-\infty}^{\tau=\infty} g(\tau) f(t-\tau) d\tau$$

and making the change of variables τ'=*t*-τ (so τ=*t*-τ' and dτ=-dτ') gives

$$(g * f)(t) = \int_{t-\tau'=-\infty}^{t-\tau'=\infty} g(t-\tau') f(\tau')(-d\tau') = \int_{\tau'=-\infty}^{\tau'=\infty} f(\tau') g(t-\tau') d\tau' = (f * g)(t).$$

[1] The variable *τ* is a "dummy" variable because it is used only during the process of integration and plays no role in the result (it disappears when the limits of integration are inserted, so it is said to be "integrated out" in the process). This means that this variable could be given any name and the integration result would be the same.

Another benefit of reversing either function prior to shifting and multiplication can be understood by considering an application in which a signal (represented in the time domain by one of the functions) is applied to a system (with time-domain characteristics represented by the other function). If you wish to determine the response of the system to that signal, it's necessary to analyze how the signal and the system interact as the signal arrives at and then moves through the system.

To visualize the interaction between a signal and a system, it may help to imagine the "signal" as the time-varying voltage output of a microphone and the "system" as a filtering circuit into which the signal is injected. Alternatively, you can picture the signal as a radar pulse that propagates across and scatters from an extended target (the system) as time passes.

If the function *g(t)* represents the signal (the output of the microphone or the shape of the radar pulse) and the function *f(t)* represents the system (the impulse response of the filter or the scattering profile of the target), the interaction as the signal arrives at, moves through, and departs from the system can be modelled by the multiplication of two functions as one "slides past" the other. The result of the interaction as a function of time is given by the convolution of the system function *f(t)* and the signal function *g(t)*.

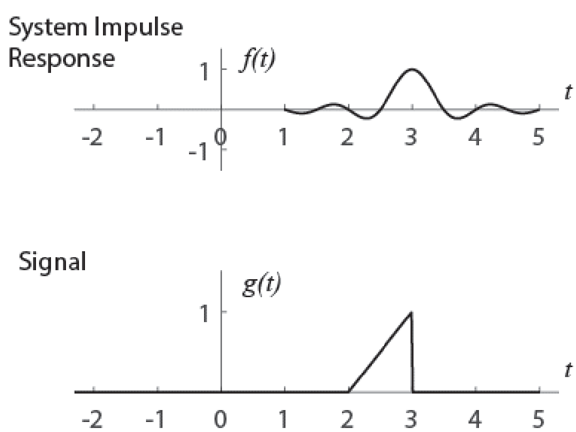


Figure 1: Time-domain functions representing a system *f(t)* and a signal *g(t)*.

Now think about how you might plot a signal (the microphone output or the radar pulse) on a graph in which the (horizontal) time axis is increasing to the right, as in Figure 1. In that case, the earliest portion of the signal is leftmost on the plot (since it occurs at smaller values of time), and the later portions of the signal appear to the right on the plot (at larger values of time). So it's the leftmost portion of the plotted time-domain signal (the beginning of the upward ramp in this case) that begins interacting with the system before the later (rightward) portions of the signal have arrived.

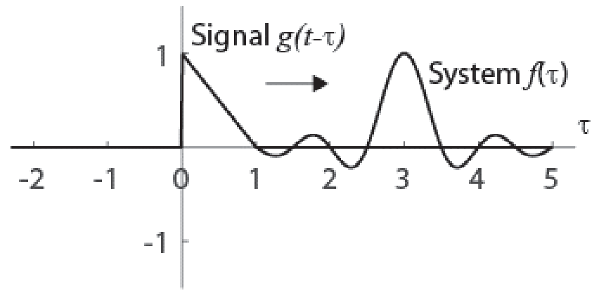


Figure 2: Representation of the signal *g(t-τ)* moving through the system *f(τ)* as *t* increases.

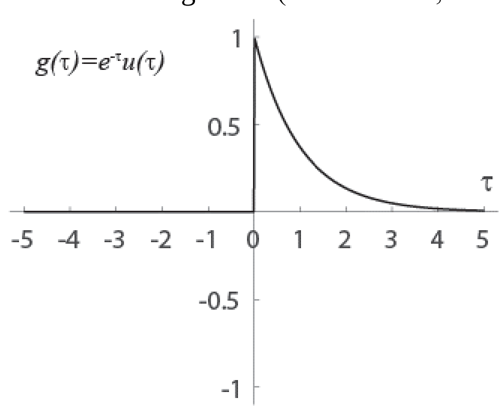
The interaction between the signal *g(t-τ)* and the system *f(τ)* is depicted in Figure 2. At the instant shown in the figure, the leading edge of the signal has just arrived at the system, so the interaction is about to begin. As time passes (that is, as the value of *t* increases), the function *g(t-τ)* will move to the right as the signal moves through the system.

Notice that by reversing the signal function, the earliest portion of the signal (the beginning of the upward ramp in this case) arrives first at the system, with later portions of the signal (the higher portions of the ramp and the step back from one to zero in this case) following at later times. But if you hadn't reversed the function representing signal, the later (rightmost) portion of the signal would have been the first to interact with the system. Remember, due to the commutativity of the convolution process, you can reverse either of the two functions and the result will be the same.

So reversing one of the functions provides the dual benefits of commutativity and correct time-domain sequencing when convolution is used for applications such as this. But if you're uncertain as to why the expression *g(t-τ)* in the convolution integral accomplishes the required flipping and time-offset of the function *g(t)*, you may find the next section helpful.

Section 3: Flipping and shifting functions

To understand the effect of the minus sign in the function *g(-τ)* and the constant *c* in *g(-τ±c)*, start by considering the exponential time-domain function $g(\tau)=e^{-\tau}u(\tau)$ shown in Figure 3. Note the small table on the right of the graph, showing several values of the independent variable and the function *g* with that variable as its argument (as in the text, $\tau=0^+$ is an instant just after $\tau=0$).



τ	g(τ)
-3	0
-2	0
-1	0
0^+	1
1	1/e
2	$1/e^2$
3	$1/e^3$

Figure 3: Time-domain function $g(\tau)=e^{-\tau}u(\tau)$.

The variable is called τ in this plot and table, but neither the shape of the plot nor the entries in the table depend on the label of that variable. In other words, an identical plot and table would result had we plotted $g(t)$ vs. t, $g(\alpha)$ vs. α, or $g(a+1)$ vs. $a+1$. But as shown in the remainder of this section, the graph can change significantly if you plot $g(-t)$ vs. t (rather than vs. $-t$) or $g(a+1)$ vs. α (rather than vs. $a+1$).

You can see an example of that in Figure 4. Look first at the two left columns in the table, labelled $\tau+1$ and $g(\tau+1)$. Note that the entries in these two columns are identical to the entries in the table in Figure 3, because as long as the entries in the left column are used as the argument of the function g, that function takes on the values shown in the center column.

$\tau+1$	$g(\tau+1)$	τ
-3	0	-4
-2	0	-3
-1	0	-2
0^+	1	-1^+
1	$1/e$	0
2	$1/e^2$	1
3	$1/e^3$	2

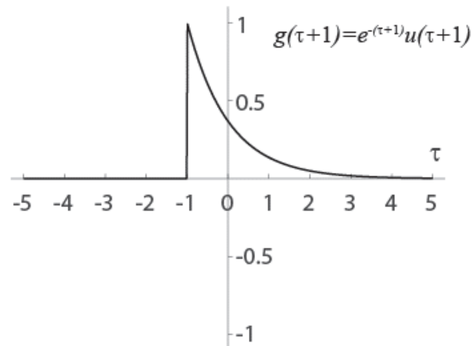


Figure 4: Offset time-domain function $g(\tau+1)= e^{-(\tau+1)}u(\tau+1)$.

Now look at the rightmost column of the table in Figure 4, which shows the values of τ corresponding to the values of $\tau+1$ in the leftmost column. If you plot $g(\tau+1)$ vs. τ (rather than $\tau+1$), you get the plot shown on the right side of Figure 4. As that plot shows, the function $g(\tau+1)$ is shifted one unit to the left (toward negative τ) when plotted vs. τ. Had we used the argument $\tau-1$ in this plot, the function $g(\tau-1)$ would have been shifted not to the left but to the right (toward positive τ) when plotted vs. τ.

So adding a positive constant to the argument of the function $g(\tau)$ shifts the plot of the function to the left, and subtracting a constant shifts the function plot to the right. But if the independent variable (τ in this case) has a minus sign in front of it, two changes occur: the function is reversed (flipped about the vertical axis), and any shift goes in the opposite direction.

To see why that's true, start by looking at the table on the left side of Figure 5. Once again, the two left columns have identical entries to the previous tables, but in this case, those columns are labelled $-\tau$ and $g(-\tau)$. The rightmost column shows the values of τ corresponding to the values of $-\tau$ in the leftmost column, and if you plot $g(-\tau)$ vs. τ (rather than $-\tau$), you get the plot shown on the right side of Figure 5. As that plot shows, the function $g(-\tau)$ is flipped about the vertical axis when plotted vs. τ.

$-\tau$	$g(-\tau)$	τ
-3	0	3
-2	0	2
-1	0	1
0^+	1	0^-
1	1/e	-1
2	$1/e^2$	-2
3	$1/e^3$	-3

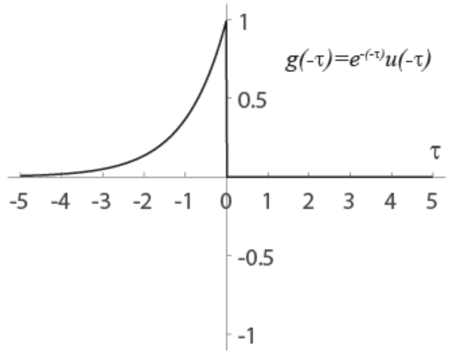


Figure 5: Reversed time-domain function $g(-\tau)=e^{-(-\tau)}u(\tau)$.

The final figure of this section (Figure 6) shows the effect of adding a positive constant to the argument of the reversed function $g(-\tau)$. Note that the entries in the two left columns are identical to the entries in Figures 3, 4, and 5, although in this case they represent $-\tau+1$ and $g(-\tau+1)$. The rightmost column shows the values of τ corresponding to the values of $-\tau+1$ in the leftmost column.

$-\tau+1$	$g(-\tau+1)$	τ
-3	0	4
-2	0	3
-1	0	2
0^+	1	1^-
1	1/e	0
2	$1/e^2$	-1
3	$1/e^3$	-2

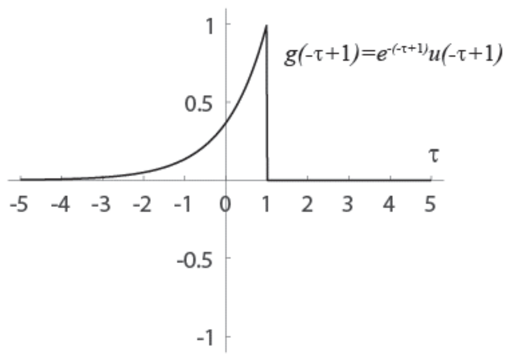


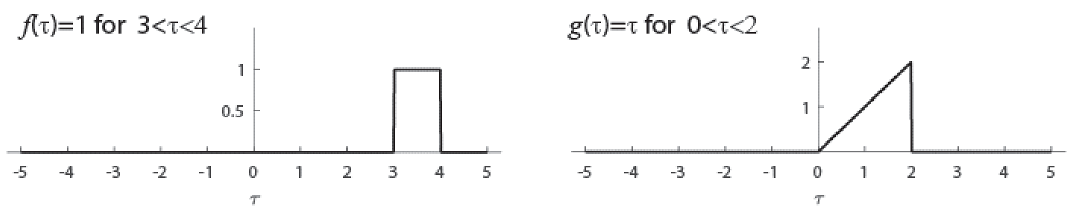
Figure 6: Reversed and offset time-domain function $g(-\tau+1)= e^{-(-\tau+1)}u(-\tau+1)$.

Plotting the values of $g(-\tau+1)$ from the center column vs. the values of τ from the rightmost column gives the graph on the right side of Figure 6. Comparing this to the graph in Figure 5 shows that the effect of adding a positive constant in this case is to shift the graph of the reversed function $g(-\tau)$ vs. τ to the right (toward positive τ). As mentioned above, that's opposite to the direction of the shift of the (unreversed) function $g(\tau)$ vs. τ when a positive constant is added to the argument.

This explains why the expression $g(t-\tau)$ appears in the convolution integral in Equation 1: it produces the reversal and time-shift of the function $g(\tau)$ needed for the convolution process. The limits of that integral are $-\infty$ to $+\infty$, but when you perform that integration, particularly in the case in which the functions being convolved are of finite duration, it's necessary to break the integral into subranges of τ, and setting the limits of integration for each subrange takes a bit of thought. That's the subject of the final section of this document.

Section 4: Graphical explanation of convolution process

As described above, the convolution of two finite-duration time-domain functions such as *f(τ)* and *g(τ)* shown in Figure 7 is performed by reversing one of the functions (chosen to be *g(τ)* in this example) and multiplying that reversed function by the other function (*f(τ)* in this case) as *g(t-τ)* "slides past" *f(τ)* over time. During that process, the overlap between the two functions changes as *g(t-τ)* moves into and then moves out of the region in which *f(τ)* is non-zero, and that means that the convolution integral can be done over subranges of *τ*. For each of those subranges, the limits of integration depend on the positions of the left and right edges of *f(τ)* and *g(t-τ)*.

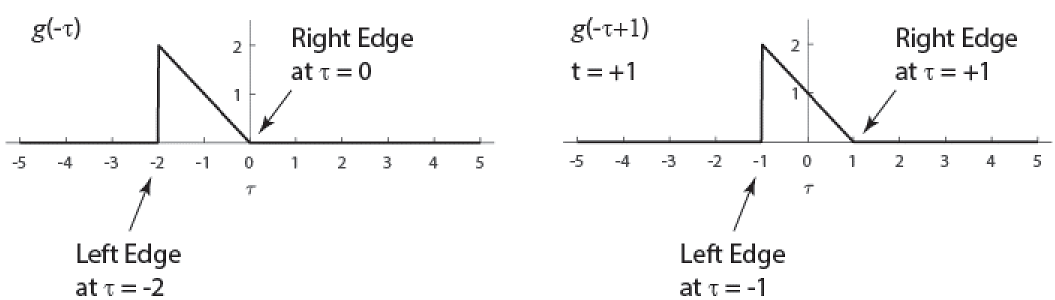


(a) (b)

Figure 7: Finite-duration time-domain functions *f(τ)* and *g(τ)*.

To see an example of how that works, look at the edges of the function *f(τ)* shown in Figure 7a. The function *f(τ)* is independent of *t* and does not move along as time *t* increases, so the left edge of the function *f(τ)* is always at *τ*=3 and the right edge of *f(τ)* is always at τ=4.

Now look at the edges of the function *g(τ)* shown in Figure 7b. Before this function is reversed and offset, this function extends from *τ*=0 to *τ*=2. But when *g(τ)* is reversed (but not offset), it extends from *τ*=-2 to *τ*=0, as shown in Figure 8a. So at time *t*=0, the left edge of *g(-τ)* is at *τ*=-2 and the right edge is at *τ*=0. But since the function *g(t-τ)* moves (rightward) along the *τ* axis as *t* increases, the left and right edges both change position over time.



(a) (b)

Figure 8: Position of left and right edges of time-domain functions *g(-τ)* and *g(-τ+1)*.

One example of that is shown in Figure 8b, in which you can see that at time $t=+1$, the left edge of $g(t-\tau)$ is positioned at $\tau=-1$ and the right edge at $\tau=+1$. So for this function, at any time t the left edge is at $\tau=t-2$ and the right edge is at $\tau=t$.

Since it's the region of overlap between function $f(\tau)$ and function $g(t-\tau)$ that contributes to the convolution integral, it's important to determine the extent of that overlap as well as the time range over which the overlap occurs. You can do that with the help of a series of plots such as those shown in Figure 9. In this figure, the position of the reversed and offset function $g(t-\tau)$ relative to the function $f(\tau)$ is shown at various times. Those times have been selected to show $g(t-\tau)$ approaching, entering, covering, and leaving the time range during which $f(\tau)$ is non-zero.

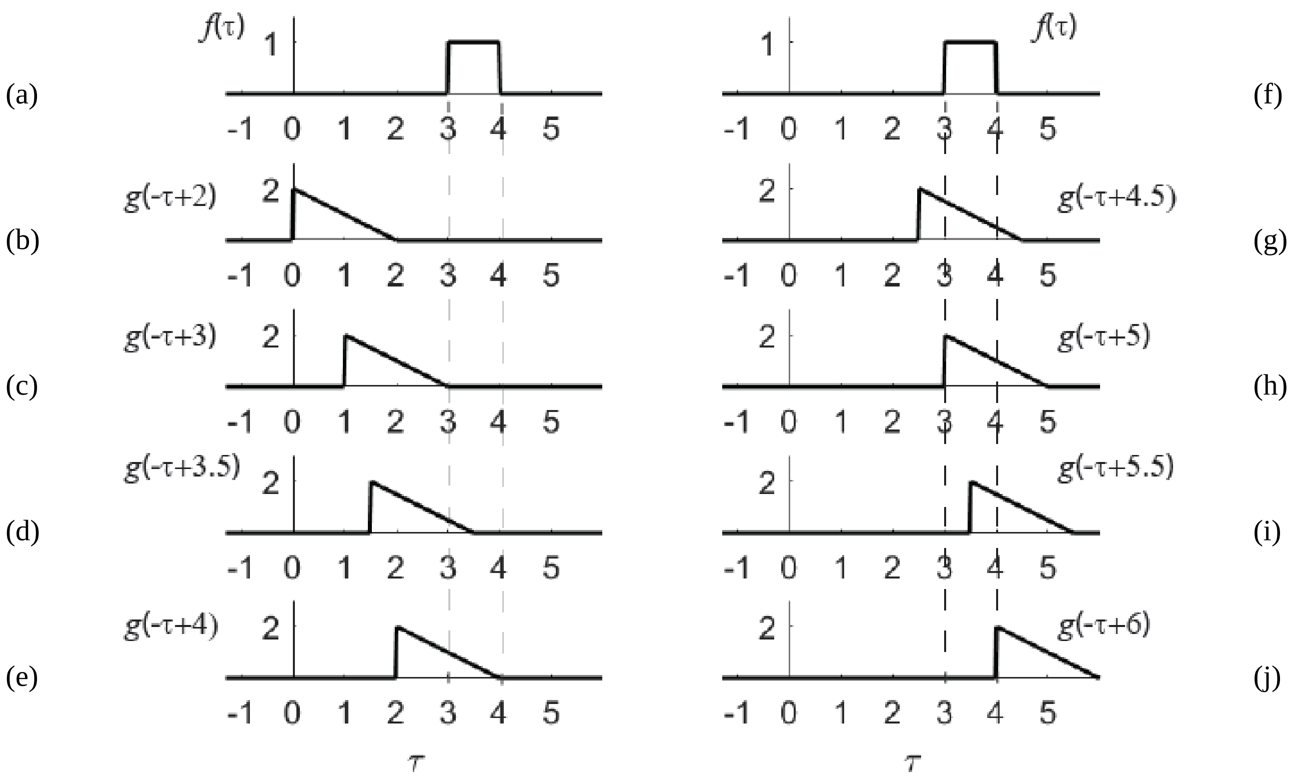


Figure 9: Overlap of $f(\tau)$ and $g(t-\tau)$ as time increases from $t=2$ to $t=6$.

The left column of Figure 9 shows the position of $g(t-\tau)$ at times t = 0, 2, 3, 3.5, and 4, and the right column shows position of $g(t-\tau)$ at times t = 4.5, 5, 5.5, and 6. At each position for which some overlap between $f(\tau)$ and of $g(t-\tau)$ occurs, the lower limit of integration is set by the left edge of either $f(\tau)$ or $g(t-\tau)$ (whichever is at larger τ) and the upper limit of integration is set by the right edge of either $f(\tau)$ or $g(t-\tau)$ (whichever is at smaller τ).

For $t<3$ (as in Figure 9 b and c), the function $g(t-\tau)$ lies entirely to the left of the function $f(\tau)$, and there is no overlap between the functions.

For $3<t<4$ (Figure 9 d and e), $g(t-\tau)$ is entering the time range in which $f(\tau)$ is non-zero. The left edge of the overlap region is set by the left edge of $f(\tau)$ at $\tau=3$, and the right edge of the overlap region is set by the right edge of $g(t-\tau)$, which is at $\tau=t$. Thus the integration limits are $\tau=3$ to $\tau=t$.

For 4<*t*<5 (as in Figure 9 g and h), the function *g(t-τ)* begins before and ends after the time range in which *f(τ)* is non-zero, so the region of overlap begins at the left edge of *f(τ)* at τ=3 and ends at the right edge of *f(τ)* at τ=4. In this case, the limits of integration are *τ*=3 to *τ*=4.

For 5<*t*<6 (as in Figure 9 i and j), the left edge of the function *g(t-τ)* has moved past the left edge of *f(τ)*, so the overlap region begins at the left edge of *g(t-τ)* at *τ*=*t*-2. The right edge of *g(t-τ)* is past the right edge of *f(τ)*, so the overlap region ends at the right edge of *f(τ)* at τ=4. Hence the limits of integration for this region are *τ*=*t*-2 to *τ*=4.

For *t*>6, the left edge of the function *g(t-τ)* has moved past the right edge of *f(τ)*, so there is no region of overlap.

With these time ranges and integrations for the regions of overlap in hand, the convolution can be determined for each subrange of τ:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau =$$

$$\int_{-\infty}^{3}(0)(t-\tau)\,d\tau = 0 \qquad t < 3$$

$$\int_{3}^{t}(1)(t-\tau)d\tau = t(t-3) - \frac{1}{2}(t^2-3^2) = \frac{t^2}{2} - 3t + \frac{9}{2} = \frac{1}{2}(t-3)^2 \qquad 3 < t < 4$$

$$\int_{3}^{4}(1)(t-\tau)d\tau = t(4-3) - \frac{1}{2}(4^2-3^2) = t - \frac{7}{2} \qquad 4 < t < 5$$

$$\int_{t-2}^{4}(1)(t-\tau)d\tau = t[4-(t-2)] - \frac{4^2-(t-2)^2}{2}] = -\frac{t^2}{2} + 4t - 6 = -\frac{(t-6)(t-2)}{2} \qquad 5 < t < 6$$

$$\int_{6}^{\infty}(0)(t-\tau)d\tau = 0 \qquad t > 6.$$

Here's a plot of the convolution as a function of time:

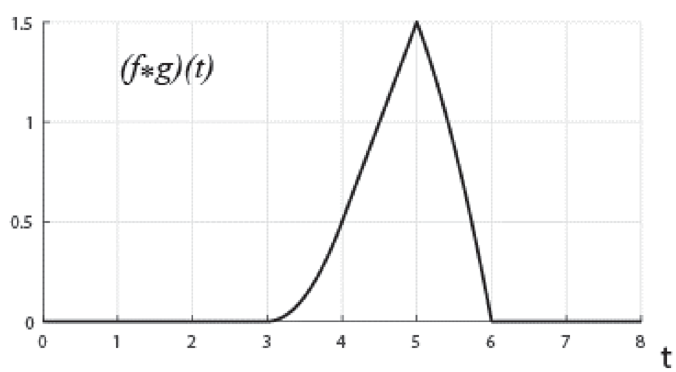


Figure 10: Result of convolution $f(t) * g(t)$.

For additional discussion and examples of convolution in the context of the Fourier transform, you may find *The Fast Fourier Transform and Its Applications* by E.O. Brigham (Pearson, 1988, ISBN 978-0133075052) helpful.