Supplemental Material for *A Student's Guide to Laplace Transforms*

Derivatives: Slopes of Sinusoidal and Real Exponential Functions

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As part of the explanation in the text of *A Student's Guide to Laplace Transforms* of why taking the derivative with respect to time of $f(t)$ prior to taking the Laplace transform has the effect of multiplying $F(s)$ by s, the fact that taking the time derivative brings down a factor of $i\omega$ from $e^{i\omega t}$ and a factor of σ from $e^{\sigma t}$. The text also mentions that taking the time derivative of a function at any time is equivalent to finding the slope in the graph of the function at that time. The purpose of this short supplemental document is to illustrate the process of finding the slope for sinusoidal and real exponential functions.

You can see how that's done for sinusoidal functions in Fig. 0.1. Recall that the slope of a function is taken to be the "rise" (vertical change in the function's value at two points) divided by the "run" (horizontal distance between those two points) in the graph of the function vs. time. The run, called Δt in this figure, is assumed to be small in order to minimize the effect of curvature (changing slope) between the two points.

In the (a) portion of Fig. 0.1, the sinusoidal function is taken as $\sin(\omega t)$, the imaginary part of the complex exponential $e^{i\omega t}$. The value of the slope at any point in time can be found by writing the time difference between the two points as Δt and using the relation $\sin(x+y) = \sin x \cos y + \cos x \sin y$:

$$
\begin{aligned}
\frac{df}{dt} &= \lim_{\Delta t \to 0} \frac{\sin[\omega(t+\Delta t)] - \sin(\omega t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\sin(\omega t)\cos(\omega \Delta t) + \cos(\omega t)\sin(\omega \Delta t) - \sin(\omega t)}{\Delta t} \\
&\approx \frac{\sin(\omega t)(1) + \cos(\omega t)(\omega \Delta t) - \sin(\omega t)}{\Delta t} \approx \frac{\cos(\omega t)(\omega \Delta t)}{\Delta t} \\
&\approx \omega \cos(\omega t)
\end{aligned}
$$

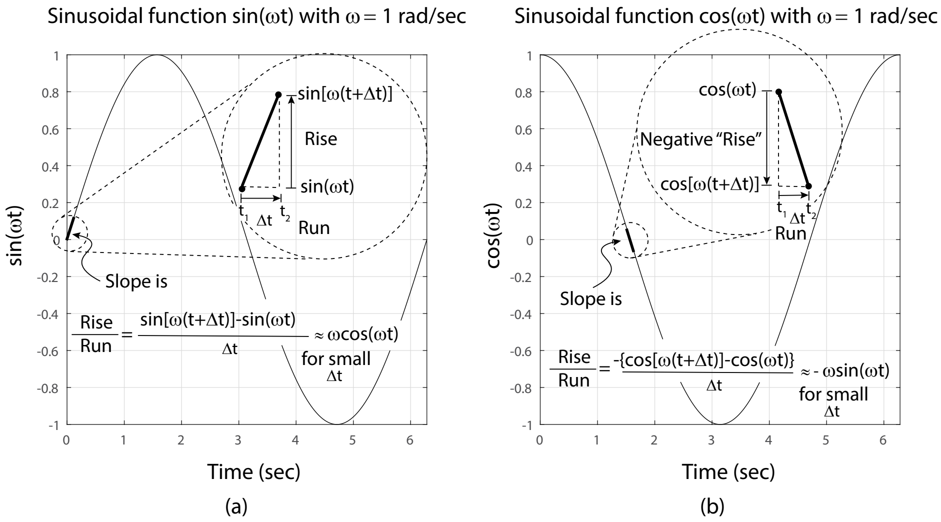

Figure 1.1 Slopes of sinusoidal functions.

in which the small-angle relations $\sin x \approx x$ and $\cos x \approx 1$ for small values of x can be used since Δt is small.

The steepest slope for this sinusoidal function occurs when $\cos(\omega t) = 1$ (such as at time $t = 0$). That maximum slope has value ω, which is reasonable since higher frequency means shorter period between peaks and valleys, and that requires steeper slopes.

In the (b) portion of Fig. 0.1, the sinusoidal function is taken as $\cos(\omega t)$, the real part of the complex exponential $e^{i\omega t}$. In this case, the value of the slope at any point in time can be found by writing the time difference between the two points as Δt and using the relation $\cos(x + y) = \cos x \cos y - \sin x \sin y$:

$$\begin{aligned}
\frac{df}{dt} &= \lim_{\Delta t \to 0} \frac{\cos[\omega(t + \Delta t)] - \cos(\omega t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\cos(\omega t)\cos(\omega \Delta t) - \sin(\omega t)\sin(\omega \Delta t) - \cos(\omega t)}{\Delta t} \\
&\approx \frac{\cos(\omega t)(1) - \sin(\omega t)(\omega \Delta t) - \cos(\omega t)}{\Delta t} \approx \frac{-\sin(\omega t)(\omega \Delta t)}{\Delta t} \\
&\approx -\omega \sin(\omega t)
\end{aligned}$$

in which the small-angle relations $\sin x \approx x$ and $\cos x \approx 1$ for small values of x can be used since Δt is small.

This explains why the change over time of a complex exponential basis function $e^{i\omega t}$ looks like this:

$$\frac{de^{i\omega t}}{dt} = \frac{d}{dt}[\cos(\omega t) + i\sin(\omega t)]$$
$$= -(\omega)\sin(\omega t) + i\omega\cos(\omega t) = i\omega e^{i\omega t}.$$

So the process of taking the time derivative of the complex exponential function $e^{i\omega t}$ brings a factor of $i\omega$ down from the exponent, and that factor is multiplied by the function itself. Hence the change in a complex-exponential basis function over time is just $i\omega$ times that same basis function.

This makes sense, because multiplying $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ by the factor of i converts the imaginary part of the function from $\sin(\omega t)$ to $\cos(\omega t)$ and the real part from $\cos(\omega t)$ to $-\sin(\omega t)$. That's exactly what's needed to convert the shape of $e^{i\omega t}$ into the shape of the slope of $e^{i\omega t}$, and multiplying by the factor of ω scales the amplitude to match the value of the slope at each point in time.

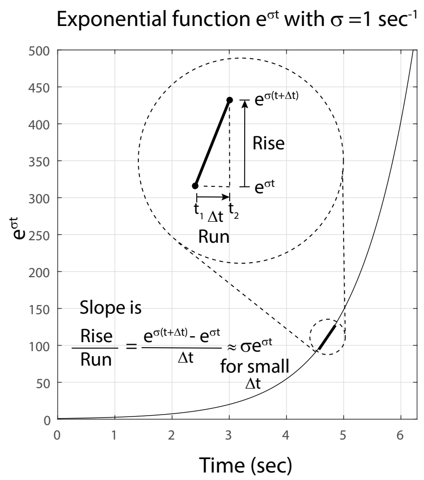

Figure 1.2 Slope of exponential function.

To appreciate the significance of this result, it helps to also consider the change over time of the real exponential function $e^{\sigma t}$. As illustrated in Fig. 0.2, that slope can be found as

$$\begin{aligned}
\frac{de^{\sigma t}}{dt} &= \lim_{\Delta t \to 0} \frac{e^{\sigma(t+\Delta t)} - e^{\sigma t}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{e^{\sigma t} e^{\sigma \Delta t} - e^{\sigma t}}{\Delta t} \approx \frac{e^{\sigma t}(1 + \sigma \Delta t) - e^{\sigma t}}{\Delta t} \\
&\approx \frac{e^{\sigma t}(\sigma \Delta t)}{\Delta t} \approx \sigma e^{\sigma t}
\end{aligned}$$

in which the relation $e^x \approx 1 + x$ for small values of x has been used.

Thus the time derivative of the real exponential function $e^{\sigma t}$ brings a factor of σ down from the exponent, and that factor is multiplied by the function itself ($e^{\sigma t}$ in this case). Just as the change in a sinusoidal basis function over time is $i\omega$ times that basis function, the change in the real exponential function $e^{\sigma t}$ over time is simply σ times that same real exponential function.

This is the reasoning behind the explanation in the text that in determining the change over time of the basis functions $e^{i\omega t}$ and the real exponential function $e^{\sigma t}$ that are used to synthesize the time-domain function $f(t)$, you find that the same basis functions are still present, scaled by $i\omega$, and the same real exponential is still present, scaled by σ.