Supplemental Material for *A Student's Guide to Laplace Transforms*

Partial Fractions

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This supplemental document is intended to help readers of *A Student's Guide to Laplace Transforms* seeking a review of partial fractions and their use in finding the inverse Laplace transform of an s-domain function $F(s)$. As mentioned in the text, one of the most common approaches for determining the inverse Laplace transform is to find the function $F(s)$ in a table of Laplace transforms and then simply to look at the corresponding time-domain function $f(t)$. But to prevent those tables from containing hundreds of entries, only reasonably simple forms of $F(s)$ are typically shown, so it's definitely worth your while to develop an understanding of methods to convert various forms of $F(s)$ into combinations of the simple forms available in tables. One of those methods is the use of partial fractions.

Partial fractions are useful when you're dealing with Laplace transforms because most s-domain functions $F(s)$ are rational functions – that is, $F(s)$ can be expressed as the ratio of polynomials in s. So you can write

$$F(s) = \frac{Num(s)}{Denom(s)},$$

and if the denominator polynomial $Denom(s)$ contains a product of two or more polynomials, or a polynomial raised to a power, or a quadratic or higher-order polynomial that can't be reduced to lower-order polynomials with real roots, then partial fractions can help.

Happily, most s-domain functions $F(s)$ are not only rational but also proper functions, which means that the order of the denominator polynomial (that is, the highest power of s in the denominator) is higher than the order of the numerator polynomial. If that's not that case, and the order of $Denom(s)$ is lower than the order of $Num(s)$, then you can use polynomial long division (or synthetic division) to convert the improper rational function into the combination of a polynomial and a proper rational function.

For example, given the improper rational function

$$F(x) = \frac{Num(x)}{Denom(x)} = \frac{2x^2 - 3x + 1}{x - 2}$$

polynomial long division can be used to convert $F(x)$ to

$$\begin{array}{r|l} & \quad\quad 2x + 1 \\ \hline x - 2) & 2x^2 - 3x + 1 \\ & -2x^2 + 4x \\ \hline & \quad\quad x + 1 \\ & \quad -x + 2 \\ \hline & \quad\quad\quad 3 \end{array} \qquad \rightarrow \qquad F(x) = (2x + 1) + \left(\tfrac{3}{x-2}\right).$$

This is worth knowing, but as you're aware if you've worked through the examples of Laplace transforms in Chapter 2 of the text, this step is not necessary for most s-domain functions $F(s)$ that are the Laplace transforms of time-domain functions $f(t)$. That's because the Laplace transform $F(s)$ approaches zero as s approaches infinity, which happens when the power of s in the denominator is larger than the power of s in the numerator. Hence $F(s)$ is a proper rational function in most cases of interest.

One important aspect of using partial fractions to decompose $F(s)$ into a combination of simpler terms is the process of factoring the denominator polynomial. To determine the factors in $Denom(s)$, remember that the factor theorem says that the expression $s - p_1$ is a factor of a given polynomial if and only if p_1 is a zero of that polynomial – that is, if plugging $s = p_1$ into the polynomial yields a value of zero. Another way of expressing that concept is to say that p_1 must be a root of the polynomial equation $Denom(s) = 0$. When dealing with Laplace transforms, the zeros of the denominator polynomial are often called the poles of $F(s)$, since at these values $F(s)$ becomes infinitely tall, as described in Section 1.5 of the text.

The discussion of partial fractions in the remainder of this document is presented in three sections. The first section deals with the case in which the denominator polynomial $Denom(s)$ can be factored into the product of two or more expressions of the form $(s - p_n)$ and the zeros p_n of $Denom(s)$ are real and different from one another. The second section covers situations in which the $Denom(s)$ polynomial cannot be reduced to first-order terms such as $s - p_n$, and the third section deals with the case in which one or more of the zeros of the denominator polynomial appear more than once, so $Denom(s)$ may contain a term such as $(s - p_1)^n$. Several on-line resources and texts with discussions of partial fractions that you may find helpful are listed at the end of this document.

Distinct Real Roots

If the denominator polynomial $Denom(s)$ can be factored into the product of linear (first-order) polynomials, $F(s)$ can be written as

$$F(s) = \frac{Num(s)}{Denom(s)} = \frac{Num(s)}{(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_N)} \tag{1.1}$$

and the roots of the polynomial equation are called "simple poles" since each appears in a first-order polynomial. In this case, these roots are "distinct" because each appears only once.

To expand $F(s)$ into partial fractions, start by writing $F(s)$ as a series of fractions in which each fraction has a constant numerator and the denominators are $s - p_1$, $s - p_2$, and so forth up to $s - p_n$:

$$\frac{Num(s)}{(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_N)} = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \frac{C_3}{s-p_3} + \cdots + \frac{C_N}{s-p_N}. \tag{1.2}$$

The numerator constants C_1, $C_2 ... C_n$ can be determined by multiplying both sides of this equation by the left-side denominator:

$$\begin{aligned} Num(s) &= \frac{[Denom(s)]C_1}{s-p_1} + \frac{[Denom(s)]C_2}{s-p_2} + \frac{[Denom(s)]C_3}{s-p_3} + \cdots + \frac{[Denom(s)]C_N}{s-p_N} \\ &= \frac{[(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_N)]C_1}{s-p_1} + \frac{[(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_N)]C_2}{s-p_2} \\ &\quad + \frac{[(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_N)]C_3}{s-p_3} + \cdots + \frac{[(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_N)]C_N}{s-p_N} \end{aligned}$$

or

$$\begin{aligned} Num(s) &= \frac{[\cancel{(s-p_1)}(s-p_2)(s-p_3)\cdots(s-p_N)]C_1}{\cancel{s-p_1}} + \frac{[(s-p_1)\cancel{(s-p_2)}(s-p_3)\cdots(s-p_N)]C_2}{\cancel{s-p_2}} \\ &\quad + \frac{[(s-p_1)(s-p_2)\cancel{(s-p_3)}\cdots(s-p_N)]C_3}{\cancel{s-p_3}} + \cdots + \frac{[(s-p_1)(s-p_2)(s-p_3)\cdots\cancel{(s-p_N)}]C_N}{\cancel{s-p_N}}. \end{aligned}$$

This equation holds for any value of s, so consider what happens if you set $s = p_1$. In that case, the factor $(s - p_1)$ is zero. Note that this factor appears in the numerator of every term on the right side of this equation except the first term, in which $s - p_1$ has canceled out. That means

$$Num(s)|_{s=p_1} = [(s - p_2)(s - p_3)\cdots(s - p_N)]|_{s=p_1} C_1 + 0 + 0 + \cdots + 0.$$

and solving for C_1 gives

$$C_1 = \frac{Num(s)|_{s=p_1}}{[(s - p_2)(s - p_3)\cdots(s - p_N)]|_{s=p_1}}. \tag{1.3}$$

Since

$$[(s - p_2)(s - p_3)\cdots(s - p_N)] = \frac{Denom}{s - p_1},$$

in some texts you'll see Eq. 1.3 written as

$$C_1 = \frac{[(s - p_1)Num(s)]|_{s=p_1}}{Denom(s)|_{s=p_1}} = [(s - p_1)F(s)]|_{s=p_1}. \tag{1.4}$$

Some students find this expression confusing, since performing the substitution $s = p_1$ makes the factor $(s - p_1)$ zero, but remember that this factor also appears in the denominator of $F(s)$. So as long as you multiply $(s - p_1)$ by $F(s)$ before substituting p_1 for s, the $s - p_1$ factors will cancel and you'll get the correct value for C_1 (the square brackets are intended to remind you to perform the multiplication before making the substitution).

The same analysis with $s = p_2$ leads to

$$C_2 = \frac{[(s - p_2)Num(s)]|_{s=p_2}}{Denom(s)|_{s=p_2}} = [(s - p_2)F(s)]|_{s=p_2} \tag{1.5}$$

and

$$C_N = \frac{[(s - p_N)Num(s)]|_{s=p_N}}{Denom(s)|_{s=p_N}} = [(s - p_N)F(s)]|_{s=p_N}. \tag{1.6}$$

Once you've seen the effect of multiplying through by the denominator in the equations shown above, you should be able to understand the reasoning behind a very quick and popular method of finding the values for the constants in the partial-fraction expansion of $F(s)$. That method is called the Heaviside "cover up" method, and it's illustrated in Figures 1.1 and 1.2.

As you can see in these figures, the trick in this technique is to cover up some terms on both sides of Eq. 1.2. To find the constant C_1, for example, you cover up the $s - p_1$ term in the denominator of the left side of the equation and the denominator of the C_1 fraction on the right side (since those terms cancel when you multiply through by the left-side denominator), and you also cover up all the other partial fractions on the right side (that is, those not involving C_1), since those terms include a factor of $s - p_1$ when you multiply through, and $s - p_1$ is zero when you set $s = p_1$. That leaves C_1 by itself on the right side of the equation, and $F(s)$ without the $s - p_1$ factor on the left side.

$$\frac{Num(s)}{(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_N)} = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \frac{C_3}{s-p_3} + \cdots + \frac{C_N}{s-p_N}$$

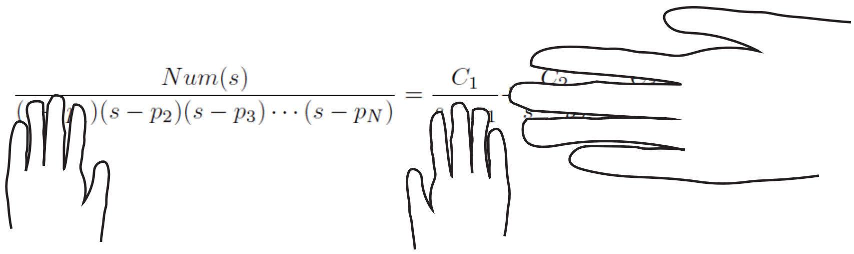

Figure 1.1 Using a version of the Heaviside "cover-up" method to find C_1.

Hence the partially covered-up equation in Figure 1.1 is a visual demonstration of Eqs. 1.3 and 1.4. But it's important to note that the equation shown in Figure 1.1 with covered-up terms and factors is only true when the value of s is set to the value of p_1.

The cover-up method applied to finding the constant C_2 is shown in Figure 1.2; in this case it's the $s - p_2$ term and all the partial fractions not involving C_2 that get covered up.

So the partially covered-up equation in Figure 1.2 is a visual demonstration of Eq. 1.5. As in the C_1 case, don't forget that the equation shown in Figure 1.2 is true only when the value of s is set to the value of the relevant pole (p_2 in this case).

In the following example, you can see partial-fraction expansion in action for simplifying $F(s)$ when there are three distinct real poles with values of 1, -2, and 4 and the numerator polynomial is $2s - 3$. The

$$\frac{Num(s)}{(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_N)} = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \frac{C_3}{s-p_3} + \cdots + \frac{C_N}{s-p_N}$$

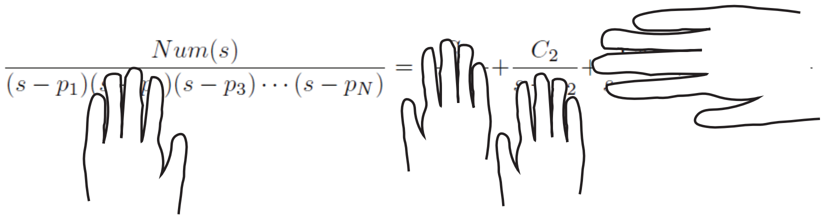

Figure 1.2 The cover-up method used to find C_2.

equation for $F(s)$ is

$$F(s) = \frac{2s-3}{(s-1)(s+2)(s-4)}$$

and the partial-fraction expansion looks like this:

$$F(s) = \frac{2s-3}{(s-1)(s+2)(s-4)} = \frac{C_1}{s-1} + \frac{C_2}{s+2} + \frac{C_3}{s-4}.$$

Inserting the values of the poles ($p_1 = 1$, $p_2 = -2$, $p_3 = 4$) into Eq. 1.4 for C_1 gives

$$C_1 = \left[\frac{2s-3}{(s+2)(s-4)}\right]\Bigg|_{s=1} = \frac{2(1)-3}{(3)(-3)} = \frac{1}{9}$$

and the other constants are

$$C_2 = \left[\frac{2s-3}{(s-1)(s-4)}\right]\Bigg|_{s=-2} = \frac{2(-2)-3}{(-3)(-6)} = -\frac{7}{18}$$

and

$$C_3 = \left[\frac{2s-3}{(s-1)(s+2)}\right]\Bigg|_{s=4} = \frac{2(4)-3}{(3)(6)} = \frac{5}{18}.$$

Hence the partial-fraction expansion for $F(s)$ in this case is

$$F(s) = \frac{2s-3}{(s-1)(s+2)(s-4)} = \left(\frac{1}{9}\right)\frac{1}{s-1} - \left(\frac{7}{18}\right)\frac{1}{s+2} + \left(\frac{5}{18}\right)\frac{1}{s-4}. \tag{1.7}$$

Each of these fractions has the form of $F(s) = \frac{1}{s-a}$ as shown in Eq. 2.10 in Section 2.2 of the text. This is the Laplace transform of the exponential time-domain function $f(t) = e^{at}$, and the linearity of the Laplace transform (discussed in Section 3.1 of the text) means that the constants multiplying each term in Eq. 1.7 also multiply the corresponding exponentials in the time domain. So

$$f(t) = \mathcal{L}^{-1}[F(s)] = \left(\frac{1}{9}\right) e^{t} - \left(\frac{7}{18}\right) e^{-2t} + \left(\frac{5}{18}\right) e^{4t}. \tag{1.8}$$

Irreducible Polynomials in Denominator

As mentioned earlier in this document, in some cases the denominator of $F(s)$ may be a polynomial that doesn't factor into real integers as in the previous section. Such polynomials are sometimes called "irreducible", although the meaning of that term depends on the nature of the factors under consideration.

For example, the polynomial $s^2 - 2$ has no integer roots, so this polynomial is irreducible over integers, but since $s^2 - 2 = (s + \sqrt{2})(s - \sqrt{2}$, this polynomial is reducible over real numbers. Likewise, the polynomial $s^2 + 3$ has no real roots, so it's irreducible over real numbers, but $s^2 + 3 = (s + \sqrt{3}i)(s - \sqrt{3}i)$, so this polynomial is reducible over complex numbers. In general, a polynomial of the form $as^2 + bs + c$ is irreducible over real numbers if (and only if) $b^2 - 4ac < 0$.

In this section, you'll see two approaches that can be used to simplify a rational polynomial with one or more irreducible terms in the denominator. The first uses the method of simultaneous equations and undetermined coefficients and the second uses complex roots.

To understand how these approaches work, consider a rational function $F(s)$ with one simple pole (p_1) and an irreducible quadratic in the denominator:

$$F(s) = \frac{Num(s)}{Denom(s)} = \frac{Num(s)}{(s - p_1)(a_d s^2 + b_d s + c_d)} \tag{1.9}$$

in which the subscript "d" in a_d, b_d, and c_d is a reminder that these coefficients apply to the polynomial in the denominator (as opposed to the coefficients of the numerator polynomial, which will be provided when you need it).

To expand $F(s)$ using partial fractions, start by writing

$$F(s) = \frac{Num(s)}{Denom(s)} = \frac{Num(s)}{(s-p_1)(a_d s^2 + b_d s + c_d)} = \frac{C_1}{s-p_1} + \frac{As+B}{a_d s^2 + b_d s + c_d}.$$

Notice that the numerator of each of the partial fractions on the right side is a polynomial with order one less than the order of the denominator polynomial. So the numerator of a partial fraction with a first-order polynomial (such as $s - p_1$) in the denominator is a zeroth-order polynomial (that is, a constant), while the numerator of a partial fraction with a second-order polynomial (such as $a_d s^2 + b_d s + c_d$) in the denominator is a first-order polynomial ($As + B$).

To determine the constants C_1, A, and B, start by multiplying through by the left-side denominator:

$$Num(s) = \frac{(s-p_1)(a_d s^2 + b_d s + c_d)C_1}{s-p_1} + \frac{(s-p_1)(a_d s^2 + b_d s + c_d)(As+B)}{a_d s^2 + b_d s + c_d}$$

or

$$Num(s) = (a_d s^2 + b_d s + c_d)C_1 + (s-p_1)(As+B). \tag{1.10}$$

If you know the numerator polynomial $Num(s)$, you can use this equation to find the constants C_1, A, and B by equating like powers of s, but it's often helpful to first determine the constant C_1 by setting $s = p_1$ (essentially the cover-up method described above).

In this case, setting $s = p_1$ gives

$$Num(s) = (a_d s^2 + b_d s + c_d)C_1 + 0 \qquad s = p_1$$

so

$$C_1 = \left[\frac{Num(s)}{a_d s^2 + b_d s + c_d}\right]\Bigg|_{s=p_1}.$$

Thus if the numerator polynomial is $Num = a_n s^2 + b_n s + c_n$, the constant C_1 is

$$C_1 = \left[\frac{a_n s^2 + b_n s + c_n}{a_d s^2 + b_d s + c_d}\right]\Bigg|_{s=p_1}. \tag{1.11}$$

To find the constants A and B, insert the $Num(s)$ polynomial into Eq. 1.10:

$$\begin{aligned} a_n s^2 + b_n s + c_n &= (a_d s^2 + b_d s + c_d)C_1 + (s-p_1)(As+B) \\ &= a_d s^2 C_1 + b_d s C_1 + c_d C_1 + sAs + sB - p_1 As - p_1 B \\ &= (a_d C_1 + A)s^2 + (b_d C_1 + B - p_1 A)s + (c_d C_1 - p_1 B) \end{aligned}$$

and then equate like powers of s. That gives

$$\begin{aligned} a_n &= a_d C_1 + A \\ b_n &= b_d C_1 + B - p_1 A \\ c_n &= c_d C_1 - p_1 B \end{aligned}$$

which means the constants are

$$\begin{aligned} A &= a_n - a_d C_1 \\ C_1 &= \frac{b_n - B + p_1 A}{b_d} \\ B &= \frac{c_d C_1 - c_n}{p_1}. \end{aligned} \tag{1.12}$$

So although all three constants C_1, A, and B can be determined using these equations, finding C_1 as shown above and then using that value to find A and B is often easier, as mentioned above.

You can see why that's true in the following example, in which the numerator polynomial is $Num(s) = 3s^2 - 2s + 4$ and the denominator of $F(s)$ contains the product of a linear term due to a simple real pole $p_1 = 3$ and the quadratic polynomial $4s^2 + 6s + 3$:

$$F(s) = \frac{3s^2 - 2s + 4}{(s-3)(4s^2 + 6s + 3)}.$$

The partial-fraction expansion of $F(s)$ can be written as

$$F(s) = \frac{3s^2 - 2s + 4}{(s-3)(4s^2 + 6s + 3)} = \frac{C_1}{s-3} + \frac{As + B}{4s^2 + 6s + 3} \tag{1.13}$$

and the constants can be found as

$$C_1 = \left. \frac{3s^2 - 2s + 4}{4s^2 + 6s + 3} \right|_{s=3} = \frac{25}{57}$$

and

$$A = a_n - a_d C_1 = 3 - (4)\frac{25}{57} = \frac{71}{57}$$

and

$$B = \frac{c_d C_1 - c_n}{p_1} = \frac{(3)\frac{25}{57} - 4}{3} = -\frac{51}{57}.$$

Inserting these values into Eq. 1.13 yields

$$F(s) = \frac{3s^2 - 2s + 4}{(s-3)(4s^2 + 6s + 3)} = \frac{\frac{25}{57}}{s-3} + \frac{\frac{71}{57}s - \frac{51}{57}}{4s^2 + 6s + 3}. \tag{1.14}$$

The Laplace transform of the first term of $F(s)$ is straightforward, as illustrated in the previous section of this document, but the second term needs a bit of work to become recognizable (that is, to allow you to determine $f(t)$ by looking for $F(s)$ in a table of Laplace transforms).

That work starts by factoring the quadratic in the denominator of the second term, and a good first step is to pull out the coefficient of the highest power of s:

$$\frac{\frac{71}{57}s - \frac{51}{57}}{4s^2 + 6s + 3} = \left(\frac{1}{4}\right) \frac{\frac{71}{57}s - \frac{51}{57}}{s^2 + \frac{3}{2}s + \frac{3}{4}}$$

and then to complete the square:

$$\left(\frac{1}{4}\right) \frac{\frac{71}{57}s - \frac{51}{57}}{s^2 + \frac{3}{2}s + \frac{3}{4}} = \left(\frac{1}{4}\right) \frac{\frac{71}{57}s - \frac{51}{57}}{(s + \frac{3}{4})^2 - \left(\frac{3}{4}\right)^2 + \frac{3}{4}} = \left(\frac{1}{4}\right) \frac{\frac{71}{57}s - \frac{51}{57}}{(s + \frac{3}{4})^2 + \frac{3}{16}}.$$

Although this expression isn't in the exact form you're likely to find in a table of Laplace transforms, if you've looked at the examples of the transform $F(s)$ of sinusoidal time-domain functions (Section 2.3) and the effect of multiplication by an exponential in the time domain (Section 3.2), the form of this equation may look somewhat familiar.

To take this equation from "somewhat familiar" to completely recognizable, it helps to make the $\frac{71}{57}s$ term in the numerator into $\frac{71}{57}(s + \frac{3}{4})$, to match the $\frac{71}{57}(s + \frac{3}{4})$ term in the denominator. That necessitates subtracting $\frac{71}{57}\left(\frac{3}{4}\right)$, like this:

$$\left(\frac{1}{4}\right) \frac{\frac{71}{57}s - \frac{51}{57}}{(s + \frac{3}{4})^2 + \frac{3}{16}} = \left(\frac{1}{4}\right) \frac{\frac{71}{57}(s + \frac{3}{4}) - \frac{71}{57}\left(\frac{3}{4}\right) - \frac{51}{57}}{(s + \frac{3}{4})^2 + \frac{3}{16}} = \left(\frac{1}{4}\right) \frac{\frac{71}{57}(s + \frac{3}{4}) - \frac{139}{76}}{(s + \frac{3}{4})^2 + \frac{3}{16}}$$

and writing this as two separate fractions gives

$$\frac{\frac{71}{57}s + \frac{51}{57}}{4s^2 + 6s + 3} = \left(\frac{1}{4}\right) \frac{\frac{71}{57}(s + \frac{3}{4})}{(s + \frac{3}{4})^2 + \frac{3}{16}} - \left(\frac{1}{4}\right) \frac{\frac{139}{76}}{(s + \frac{3}{4})^2 + \frac{3}{16}}.$$

Gathering the leading constants makes this

$$\frac{\frac{71}{57}s + \frac{51}{57}}{4s^2 + 6s + 3} = \left(\frac{71}{228}\right) \frac{s + \frac{3}{4}}{(s + \frac{3}{4})^2 + \frac{3}{16}} - \left(\frac{139}{304}\right) \frac{1}{(s + \frac{3}{4})^2 + \frac{3}{16}}$$

which can be compared to the Laplace transform $F(s)$ of time-domain exponentially weighted cosine and sine functions:

$$\mathcal{L}[e^{at}\cos(\omega_1 t)] = \frac{s - a}{(s - a) + \omega_1^2}$$

and

$$\mathcal{L}[e^{at}\sin(\omega_1 t)] = \frac{\omega_1}{(s-a)+\omega_1^2}.$$

So in this case $a = -\frac{3}{4}$ and $\omega_1^2 = \frac{3}{16}$, so $\omega_1 = \frac{\sqrt{3}}{4}$ for both terms. The first term (with $s + \frac{3}{4}$ in the numerator) is ready to be inverse-Laplace-transformed into the time domain, but the second term needs an ω_1 in the numerator, so a bit of adjustment to the leading constant is needed. Doing that makes the previous expression

$$\frac{\frac{71}{57}s + \frac{51}{57}}{4s^2+6s+3} = \left(\frac{71}{228}\right)\frac{s+\frac{3}{4}}{(s+\frac{3}{4})^2+\frac{3}{16}} - \left(\frac{139}{76\sqrt{3}}\right)\frac{\frac{\sqrt{3}}{4}}{(s+\frac{3}{4})^2+\frac{3}{16}}$$

which is ready to be inserted into Eq. 1.14:

$$F(s) = \frac{3s^2-2s+4}{(s-3)(4s^2+6s+3)} = \frac{\frac{25}{57}}{s-3} + \left(\frac{71}{228}\right)\frac{s+\frac{3}{4}}{(s+\frac{3}{4})^2+\frac{3}{16}} - \left(\frac{139}{76\sqrt{3}}\right)\frac{\frac{\sqrt{3}}{4}}{(s+\frac{3}{4})^2+\frac{3}{16}}.$$

and the inverse Laplace transform gives the time-domain function $f(t)$:

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}\left[\frac{\frac{25}{57}}{s-3}\right] + \mathcal{L}^{-1}\left[\left(\frac{71}{228}\right)\frac{s+\frac{3}{4}}{(s+\frac{3}{4})^2+\frac{3}{16}}\right] + \mathcal{L}^{-1}\left[\left(-\frac{139}{76\sqrt{3}}\right)\frac{\frac{\sqrt{3}}{4}}{(s+\frac{3}{4})^2+\frac{3}{16}}\right] \\ &= \frac{25}{57}e^{3t} + \frac{71}{228}e^{-\frac{3}{4}t}\cos\left(\frac{\sqrt{3}}{4}t\right) - \frac{139}{76\sqrt{3}}e^{-\frac{3}{4}t}\sin\left(\frac{\sqrt{3}}{4}t\right) \end{aligned}$$

or writing the coefficients using decimals rather than factions:

$$f(t) = 0.4386e^{3t} + 0.3114e^{-\frac{3}{4}t}\cos\left(\frac{\sqrt{3}}{4}t\right) - 1.056e^{-\frac{3}{4}t}\sin\left(\frac{\sqrt{3}}{4}t\right). \tag{1.15}$$

The fact that a quadratic polynomial may be irreducible over the real numbers but reducible over complex numbers suggests an alternative approach to solving this type of problem. That approach involves factoring a denominator polynomial using complex roots, and to see how that works, note that a quadratic polynomial as^2+bs+c can be factored as by writing a quadratic equation:

$$as^2 + bs + c = a(s-r_1)(s-r_2) = 0$$

in which r_1 and r_2 are the (possibly complex) roots of this equation, also called the zeros of the function $as^2 + bs + c$. Note that when the leading coefficient a (that is, the coefficient of the highest power of s) is not one, it's necessary to multiply this coefficient by the product of the two terms containing the roots of the quadratic equation ($s - r_1$ and $s - r_2$).

To find the roots of the quadratic equation, you can use the quadratic formula

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

and if $b^2 - 4ac < 0$, these roots will be complex – that is, they will contain an imaginary part, since the square root of a negative number is imaginary. Also, the second root will be the complex conjugate of the first because the imaginary parts must have opposite signs due to the $\pm$ sign in the quadratic formula.

Since these roots represent two of the poles of the function $F(s)$, it makes sense to call them p and p^*, and since they may be complex, they can be written as

$$p = \mathrm{Re}(p) + i\ \mathrm{Im}(p)$$
$$p^* = \mathrm{Re}(p) - i\ \mathrm{Im}(p)$$

in which the asterisk indicates the complex conjugate.

If the function $F(s)$ has one simple real pole p_1 and an irreducible polynomial (over the real numbers) in its denominator, Eq. 1.9 can be factored as

$$F(s) = \frac{Num(s)}{Denom(s)} = \frac{Num(s)}{(s-p_1)(a_d s^2 + b_d s + c_d)} = \frac{Num(s)}{(s-p_1)a_d(s-p_2)(s-p_2^*)}.$$

Expanding $F(s)$ using partial fractions now looks like this:

$$\frac{Num(s)}{(s-p_1)a_d(s-p_2)(s-p_2^*)} = \frac{C_1}{(s-p_1)} + \frac{C_2}{(s-p_2)} + \frac{C_3}{(s-p_1^*)}$$

and multiplying through by the denominator of the left side gives

$$Num(s) = a_d(s-p_2)(s-p_2^*)C_1 + (s-p_1)a_d(s-p_2)^* C_2 + (s-p_1)a_d(s-p_2)C_3.$$

Thus the coefficients C_1, C_2, and C_3 can be determined using the same approach shown in the first section of this document for simple real roots: solving simultaneous equations (and/or using the cover-up method). Those methods give

$$C_1 = \frac{[(s-p_1)Num(s)]|_{s=p_1}}{Denom(s)|_{s=p_1}} = [(s-p_1)F(s)]|_{s=p_1}$$

$$C_2 = \frac{[(s-p_2)Num(s)]|_{s=p_2}}{Denom(s)|_{s=p_2}} = [(s-p_2)F(s)]|_{s=p_2}.$$

$$C_3 = \frac{[(s-p_2^*)Num(s)]|_{s=p_2^*}}{Denom(s)|_{s=p_2^*}} = [(s-p_2^*)F(s)]|_{s=p_2^*}.$$

To see how this works, you can apply this approach to the example worked above. $F(s)$ in that case is given by

$$F(s) = \frac{3s^2-2s+4}{(s-3)(4s^2+6s+3)},$$

and using the quadratic formula to find the roots of the polynomial $4s^2+6s+3$ gives

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-6 \pm \sqrt{(6)^2-4(4)(3)}}{2(4)} = \frac{-6}{8} \pm \frac{\sqrt{-12}}{8} = -\frac{3}{4} \pm \frac{\sqrt{3}}{4}i$$

so $p_2 = -\frac{3}{4} + \frac{\sqrt{3}}{4}i$ and $p_2^* = -\frac{3}{4} - \frac{\sqrt{3}}{4}i$.

With these roots, along with the real root $p_1 = 3$, the partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{3s^2-2s+4}{(s-3)(4s^2+6s+3)} = \frac{C_1}{s-3} + \frac{C_2}{s-\left(-\frac{3}{4}+\frac{\sqrt{3}}{4}i\right)} + \frac{C_3}{s-\left(-\frac{3}{4}-\frac{\sqrt{3}}{4}i\right]}. \tag{1.16}$$

Solving for the constants C_1, C_2, and C_3 gives

$$C_1 = \left.\frac{3s^2-2s+4}{4s^2+6s+3}\right|_{s=3} = \frac{25}{57} = 0.4386$$

and

$$C_2 = \left.\frac{3s^2-2s+4}{(s-3)(4)\left[s-\left(-\frac{3}{4}-\frac{\sqrt{3}}{4}i\right)\right]}\right|_{s=-\frac{3}{4}+\frac{\sqrt{3}}{4}i} = 0.1557 + 0.5280i$$

and

$$C_3 = \left.\frac{3s^2-2s+4}{(s-3)(4)\left[s-\left(-\frac{3}{4}+\frac{\sqrt{3}}{4}i\right)\right]}\right|_{s=-\frac{3}{4}-\frac{\sqrt{3}}{4}i} = 0.1557 - 0.5280i$$

and inserting these values into Eq. 1.16 makes $F(s)$

$$F(s) = \frac{3s^2-2s+4}{(s-3)(4s^2+6s+3)} = \frac{0.4386}{s-3} + \frac{0.1557+0.5280i}{s-\left(-\frac{3}{4}+\frac{\sqrt{3}}{4}i\right)} + \frac{0.1557-0.5280i}{s-\left(-\frac{3}{4}-\frac{\sqrt{3}}{4}i\right)}.$$

Although the complex quantities in the second and third terms may make this expression look quite different from the types of $F(s)$ you've seen before, note that the denominators are of the form $s-a$, albeit with complex a. Note also that the numerators of these two terms are just constants, although these constants are complex. So the inverse Laplace transform of $F(s)$ is straightforward:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{0.4386}{s-3}\right] + \mathcal{L}^{-1}\left[\frac{0.1557+0.5280i}{s-\left(-\frac{3}{4}+\frac{\sqrt{3}}{4}i\right)}\right] + \mathcal{L}^{-1}\left[\frac{0.1557-0.5280i}{s-\left(-\frac{3}{4}-\frac{\sqrt{3}}{4}i\right)}\right]$$

$$= 0.4386e^{3t} + (0.1557+0.5280i)e^{\left(-\frac{3}{4}+\frac{\sqrt{3}}{4}i\right)t} + (0.1557-0.5280i)e^{\left(-\frac{3}{4}-\frac{\sqrt{3}}{4}i\right)t}.$$

This expression for $f(t)$ is correct, but it can be put into more-familiar form by writing the complex leading constants in the second and third terms in polar form – that is, using the magnitude $|s|$ and phase ϕ of the complex quantity s.

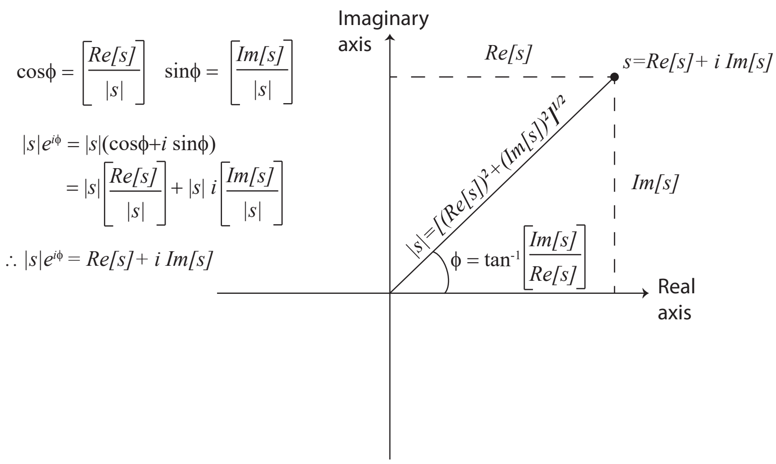

Figure 1.3 The polar form of complex number s.

You can see how that works in Fig. 1.3, which illustrates why the expression $s = Re[s] + Im[s]i$ is equivalent to $|s|e^{i\phi}$. In this case, the magnitude is given by

$$|0.1557{+}0.5280i| = \sqrt{(Re[s])^2 + (Im[s])^2} = \sqrt{(0.1557)^2 + (0.5280)^2} = 0.55048$$

and the phase is

$$\phi(0.1557+0.5280i) = \tan^{-1}\frac{Im[s]}{Re[s]} = \tan^{-1}\frac{0.5280}{0.1557} = 73.57°.$$

Inserting these values into the expression shown above for $f(t)$ gives

$$\begin{aligned} f(t) &= 0.4386e^{3t} + 0.55048e^{i73.57^\circ} e^{\left(-\frac{3}{4}+\frac{\sqrt{3}}{4}i\right)t} + 0.55048e^{-i73.57^\circ} e^{\left(-\frac{3}{4}-\frac{\sqrt{3}}{4}i\right)t} \\ &= 0.4386e^{3t} + 0.55048e^{-\frac{3}{4}t}\left(e^{i(73.57^\circ+\frac{\sqrt{3}}{4}t)} + e^{-i(73.57^\circ+\frac{\sqrt{3}}{4}t)}\right) \\ &= 0.4386e^{3t} + 0.55048e^{-\frac{3}{4}t}[2\cos(73.57^\circ + \frac{\sqrt{3}}{4}t)] \\ &= 0.4386e^{3t} + 1.10096e^{-\frac{3}{4}t}[\cos(73.57^\circ + \frac{\sqrt{3}}{4}t)]. \end{aligned}$$

Using the identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$ makes this

$$\begin{aligned} f(t) &= 0.4386e^{3t} + 1.10096e^{-\frac{3}{4}t}[\cos(73.57^\circ)\cos(\frac{\sqrt{3}}{4}t) - \sin(73.57^\circ)\sin(\frac{\sqrt{3}}{4}t)] \\ &= 0.4386e^{3t} + 1.10096e^{-\frac{3}{4}t}[0.28284\cos(\frac{\sqrt{3}}{4}t) - 0.95916\sin(\frac{\sqrt{3}}{4}t)] \\ &= 0.4386e^{3t} + 0.3114e^{-\frac{3}{4}t}\cos(\frac{\sqrt{3}}{4}t) - 1.056e^{-\frac{3}{4}t}\sin(\frac{\sqrt{3}}{4}t) \end{aligned}$$

in agreement with the result of using the simultaneous-equation approach with real roots shown above (Eq. 1.15).

You may encounter a situation in which the denominator of $F(s)$ contains two irreducible quadratic polynomials rather than one simple pole and one quadratic polynomial. So instead of Eq. 1.9, $F(s)$ will look like this:

$$F(s) = \frac{Num(s)}{Denom(s)} = \frac{Num(s)}{(a_{d1}s^2 + b_{d1}s + c_{d1})(a_{d2}s^2 + b_{d2}s + c_{d2})} \tag{1.17}$$

in which the subscript "d1" refer to the denominator's first quadratic and "d2" refer to the denominator's second quadratic.

With two quadratic (second-order) polynomials in the denominator, $F(s)$ can be simplified using a partial-fraction expansion with first-order polynomials in the numerators of both fractions:

$$\frac{Num(s)}{(a_{d1}s^2 + b_{d1}s + c_{d1})(a_{d2}s^2 + b_{d2}s + c_{d2})} = \frac{As + B}{a_{d1}s^2 + b_{d1}s + c_{d1}} + \frac{Cs + D}{a_{d2}s^2 + b_{d2}s + c_{d2}}$$

and multiplying both sides by the denominator of the left side gives

$$\begin{aligned} Num(s) =& (a_{d2}s^2 + b_{d2}s + c_{d2})(As + B) + (a_{d1}s^2 + b_{d1}s + c_{d1})(Cs + D) \\ &= a_{d2}As^3 + b_{d2}As^2 + c_{d2}As + a_{d2}Bs^2 + b_{d2}Bs + c_{d2}B \\ &\quad + a_{d1}Cs^3 + b_{d1}Cs^2 + c_{d1}Cs + a_{d1}Ds^2 + b_{d1}Ds + c_{d1}D \\ &= (a_{d2}A + a_{d1}C)s^3 + (a_{d2}B + b_{d2}A + b_{d1}C + a_{d1}D)s^2 \\ &\quad + (b_{d2}B + c_{d2}A + c_{d1}C + b_{d1}D)s + (c_{d2}B + c_{d1}D). \end{aligned}$$

Again writing the numerator polynomial as $Num = a_n s^2 + b_n s + c_n$ and equating like powers of s makes this

$$\begin{aligned} 0 &= a_{d2}A + a_{d1}C \\ a_n &= a_{d2}B + b_{d2}A + b_{d1}C + a_{d1}D \\ b_n &= b_{d2}B + c_{d2}A + c_{d1}C + b_{d1}D \\ c_n &= c_{d2}B + c_{d1}D \end{aligned}$$

These simultaneous equations can be solve using matrix algebra:

$$\begin{pmatrix} a_{d2} & 0 & a_{d1} & 0 \\ b_{d2} & a_{d2} & b_{d1} & a_{d1} \\ c_{d2} & b_{d2} & c_{d1} & b_{d1} \\ 0 & c_{d2} & 0 & c_{d1} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ a_n \\ b_n \\ c_n \end{pmatrix}$$

so the constants are

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} a_{d2} & 0 & a_{d1} & 0 \\ b_{d2} & a_{d2} & b_{d1} & a_{d1} \\ c_{d2} & b_{d2} & c_{d1} & b_{d1} \\ 0 & c_{d2} & 0 & c_{d1} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ a_n \\ b_n \\ c_n \end{pmatrix}.$$

In all of the cases discussed to this point, the roots of the denominator polynomials have been distinct, occurring only once in the denominator. The next section shows you how to handle cases involving repeated roots.

Repeated Roots

When a root of the denominator polynomial equation has a multiplicity greater than one (that is, when a factor such as $s^3 = (s)(s)(s)$ or a quantity such as $(s-a)^2 = (s-a)(s-a)$ appears in the denominator), $F(s)$ is said to have "repeated roots". The presence of repeated roots means that additional fractions must be included in the partial-fraction expansion, as explained below.

Consider the case in which $F(s)$ has one simple pole (p_1) and one repeated pole (p_2):

$$F(s) = \frac{Num(s)}{Denom(s)} = \frac{Num(s)}{(s-p_1)(s-p_2)^r}$$

in which the repeated root has a multiplicity of r. Expanding $F(s)$ using partial fractions looks like this:

$$\frac{Num(s)}{(s-p_1)(s-p_2)^r} = \frac{C_1}{s-p_1} + \frac{C_{21}}{s-p_2} + \frac{C_{22}}{(s-p_2)^2} + \frac{C_{23}}{(s-p_1)^3} + \cdots + \frac{C_{2r}}{(s-p_1)^r}$$

Notice that this expansion includes one fraction with denominator of $(s-p_2)^r$ along with additional fractions with denominators of all lower powers, so the exponents of the $s-p_2$ terms range from 1 to r. Notice also that the constants now have two sub-indices, with the first digit representing the pole number (2 in the case shown below) and the second digit representing the power of the repeated factor.

So if $r = 2$, the partial expansion of $F(s)$ is

$$\frac{Num(s)}{(s-p_1)(s-p_2)^2} = \frac{C_1}{s-p_1} + \frac{C_{21}}{s-p_2} + \frac{C_{22}}{(s-p_2)^2}$$

and multiplying through by the denominator of the left side gives

$$\begin{aligned} Num(s) &= \frac{(s-p_1)(s-p_2)^2 C_1}{s-p_1} + \frac{(s-p_1)(s-p_2)^2 C_{21}}{s-p_2} + \frac{(s-p_1)(s-p_2)^2 C_{22}}{(s-p_2)^2} \\ &= (s-p_2)^2 C_1 + (s-p_1)(s-p_2)C_{21} + (s-p_1)C_{22}. \end{aligned}$$

Just as in the case of simple poles discussed above, the C_1 constant can be found by setting $s = p_1$:

$$\begin{aligned} Num(s) &= (s-p_2)^2 C_1 + (s-p_1)(s-p_2)C_{21} + (s-p_1)C_{22} \\ &= (s-p_2)^2 C_1 + 0 + 0 \qquad \text{(for } s = p_1\text{)} \end{aligned}$$

so

$$C_1 = \left.\frac{Num(s)}{(s-p_2)^2}\right|_{s=p_1}. \tag{1.18}$$

Likewise, the C_{22} constant can be found by setting $s = p_2$:

$$\begin{aligned} Num(s) &= (s-p_2)^2 C_1 + (s-p_1)(s-p_2)C_{21} + (s-p_1)C_{22} \\ &= 0 + 0 + (s-p_1)C_{22} \qquad \text{(for } s = p_2\text{)} \end{aligned}$$

so

$$C_{22} = \left.\frac{Num(s)}{s-p_1}\right|_{s=p_2}. \tag{1.19}$$

However, to find the constants for all the terms with lower powers of the repeated factor, it's necessary to write out the simultaneous equations:

$$\begin{aligned}Num(s) &= (s-p_2)^2C_1 + (s-p_1)(s-p_2)C_{21} + (s-p_1)C_{22}\\ &= s^2C_1 - 2sp_2C_1 + p_2^2C_1 + s^2C_{21} - sp_2C_{21} - sp_1C_{21} + p_1p_2C_{21} + sC_{22} - p_1C_{22}\\ &= (C_1 + C_{21})s^2 + (-2p_2C_1 - p_2C_{21} - p_1C_{21} + C_{22})s + (p_2^2C_1 + p_1p_2C_{21} - p_1C_{22}).\end{aligned}$$

So if the numerator polynomial is $Num(s) = a_ns^2 + b_ns + c_n$, equating equal powers of s gives

$$\begin{aligned}a_n &= C_1 + C_{21}\\ b_n &= -2p_2C_1 - p_2C_{21} - p_1C_{21} + C_{22}\\ c_n &= p_2^2C_1 + p_1p_2C_{21} - p_1C_{22}.\end{aligned} \tag{1.20}$$

As in the previous cases, the constants can be found using matrix algebra:

$$\begin{pmatrix}1 & 1 & 0\\ -2p_2 & -(p_1+p_2) & 1\\ p_2^2 & p_1p_2 & -p_1\end{pmatrix}\begin{pmatrix}C_1\\ C_{21}\\ C_{22}\end{pmatrix} = \begin{pmatrix}a_n\\ b_n\\ c_n\end{pmatrix}$$

and

$$\begin{pmatrix}C_1\\ C_{21}\\ C_{22}\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\ -2p_2 & -(p_1+p_2) & 1\\ p_2^2 & p_1p_2 & -p_1\end{pmatrix}^{-1}\begin{pmatrix}a_n\\ b_n\\ c_n\end{pmatrix}. \tag{1.21}$$

But since C_1 is known, you can also use the first equation in the group Eq. 1.20 to quickly find C_{21}:

$$C_{21} = a_n - C_1. \tag{1.22}$$

You can see these equations at work in the following example, in which $F(s)$ has one simple root with $p_1 = 3$ and one repeated root with $p_2 = -2$ with a multiplicity of two (so $r = 2$). Thus $F(s)$ is given by the equation

$$F(s) = \frac{3s^2 - 2s + 4}{(s-3)(s+2)^2}. \tag{1.23}$$

The constant C_1 can be found using Eq. 1.18:

$$\begin{aligned}C_1 = \left.\frac{Num(s)}{(s-p_2)^2}\right|_{s=p_1} &= \left.\frac{3s^2 - 2s + 4}{(s-p_2)^2}\right|_{s=p_1}\\ &= \frac{3(3)^2 - 2(3) + 4}{[3-(-2)]^2} = \frac{25}{25} = 1\end{aligned}$$

and Eq. 1.19 gives C_{22} as

$$C_{22} = \frac{Num(s)}{s-p_1}\bigg|_{s=p_2} = \frac{3s^2-2s+4}{s-p_1}\bigg|_{s=p_2}$$
$$= \frac{3(-2)^2-2(-2)+4}{-2-3} = \frac{20}{-5} = -4.$$

. That leaves only C_{21}, which can be found using Eq. 1.22:

$$C_{21} = a_n - C_1 = 3 - 1 = 2.$$

Alternatively, if you have an easy way to do the required matrix inversion, Eq. 1.21 returns all three constants at once:

$$\begin{pmatrix} C_1 \\ C_{21} \\ C_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -2(-2) & -(3-2) & 1 \\ (-2)^2 & (3)(-2) & -3 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

which agrees with the results shown above.

Here's an alternative approach to finding the constants of all of the partial fractions involving repeated roots. That approach uses the equation

$$C_{2n} = \frac{1}{(r-n)!}\frac{d^{r-n}}{ds^{r-n}}\left[(s-p_2)^r F(s)\right]\bigg|_{s=p_2}. \qquad (1.24)$$

So if $r = 2$, then

$$C_{21} = \frac{1}{(2-1)!}\frac{d^{2-1}}{ds^{2-1}}\left[(s-p_2)^2 F(s)\right]\bigg|_{s=p_2}$$
$$= \frac{d}{ds}\left[(s-p_2)^2 F(s)\right]\bigg|_{s=p_2}$$

and

$$C_{22} = \frac{1}{(2-2)!}\frac{d^{2-2}}{ds^{2-2}}\left[(s-p_2)^2 F(s)\right]\bigg|_{s=p_2}$$
$$= \left[(s-p_2)^2 F(s)\right]|_{s=p_2}$$

since $0! = 1$ and $\frac{d^0}{ds^0}[(s-p_2)^2F(s)] = (s-p_2)^2F(s)$.

Here's how the derivative equation (Eq. 1.24) is used to find C_{21}, and C_{22} for $F(s)$ given by Eq. 1.23 above. For C_{21}, the derivative equation gives

$$\begin{aligned}
C_{21} &= \frac{1}{(2-1)!}\frac{d^{2-1}}{ds^{2-1}}\left[(s-p_2)^2F(s)\right]\Bigg|_{s=p_2}\\
&= \frac{d}{ds}\left[(s-p_2)^2F(s)\right]\Bigg|_{s=p_2}\\
&= \frac{d}{ds}\left[\frac{(s-p_2)^2(3s^2-2s+4)}{(s-p_1)(s-p_2)^2}\right]\Bigg|_{s=p_2}\\
&= \frac{d}{ds}\left[(3s^2-2s+4)(s-p_1)^{-1}\right]\Bigg|_{s=p_2}\\
&= \left[\frac{6s-2}{s-p_1}+\frac{(-1)(3s^2-2s+4)}{(s-p_1)^2}\right]\Bigg|_{s=p_2}.
\end{aligned}$$

Inserting values $p_1 = 3$ and $p_2 = -2$ gives

$$\begin{aligned}
C_{21} &= \left[\frac{6(-2)-2}{-2-3}+\frac{(-1)[3(-2)^2-2(-2)+4]}{(-2-3)^2}\right]\\
&= \frac{-14}{-5}+\frac{-20}{25} = 2.
\end{aligned}$$

For C_{22}, the derivative equation gives

$$\begin{aligned}
C_{22} &= \left[(s-p_2)^2F(s)\right]\Bigg|_{s=p_2}\\
&= \left[(s-p_2)^2\left(\frac{3s^2-2s+4}{(s-p_1)(s-p_2)^2}\right)\right]\Bigg|_{s=p_2}\\
&= (1)\left[\frac{3s^2-2s+4}{s-p_1}\right]\Bigg|_{s=p_2},
\end{aligned}$$

and inserting values $p_1 = 3$ and $p_2 = -2$ gives

$$\begin{aligned}
C_{22} &= \left[\frac{3(-2)^2-2(-2)+4}{-2-3}\right]\\
&= \frac{20}{-5} = -4.
\end{aligned}$$

The same techniques can be used to find the partial-fraction expansion of rational functions with repeated roots of higher multiplicity, although the calculations become a bit more tedious.

To see an example of that, consider a rational function with one simple root and one repeated root with multiplicity $r = 3$. In that case, $F(s)$ is give by

$$\frac{Num(s)}{(s-p_1)(s-p_2)^3} = \frac{C_1}{s-p_1} + \frac{C_{21}}{s-p_2} + \frac{C_{22}}{(s-p_2)^2} + \frac{C_{23}}{(s-p_2)^3}$$

and multiplying through by the denominator of the left side gives

$$\begin{aligned} Num(s) &= \frac{(s-p_1)(s-p_2)^3 C_1}{s-p_1} + \frac{(s-p_1)(s-p_2)^3 C_{21}}{s-p_2} + \frac{(s-p_1)(s-p_2)^3 C_{22}}{(s-p_2)^2} + \frac{(s-p_1)(s-p_2)^3 C_{23}}{(s-p_2)^3} \\ &= (s-p_2)^3 C_1 + (s-p_1)(s-p_2)^2 C_{21} + (s-p_1)(s-p_2)C_{22} + (s-p_1)C_{23}. \end{aligned}$$

Just as in the $r = 2$ case, the C_1 constant can be found by setting $s = p_1$:

$$\begin{aligned} Num(s) &= (s-p_2)^3 C_1 + (0)(s-p_2)^2 C_{21} + (0)(s-p_2)C_{22} + (0)C_{23} \\ &= (s-p_2)^3 C_1 \qquad \text{(for } s = p_1\text{)} \end{aligned}$$

so

$$C_1 = \left. \frac{Num(s)}{(s-p_2)^3} \right|_{s=p_1}. \tag{1.25}$$

In this $r = 3$ case, it's the C_{23} constant that can be found by setting $s = p_2$:

$$\begin{aligned} Num(s) &= (0)^3 C_1 + (s-p_1)(0)^2 C_{21} + (s-p_1)(0)C_{22} + (s-p_1)C_{23} \\ &= (s-p_1)C_{23} \qquad \text{(for } s = p_2\text{)} \end{aligned}$$

so

$$C_{23} = \left. \frac{Num(s)}{s-p_1} \right|_{s=p_2}. \tag{1.26}$$

But to determine C_{21} and C_{22}, it's necessary to use

$$\begin{aligned} Num(s) &= (s-p_2)^3 C_1 + (s-p_1)(s-p_2)^2 C_{21} + (s-p_1)(s-p_2)C_{22} + (s-p_1)C_{23} \\ &= (s^3 - 3s^2 p_2 + 3sp_2^2 - p_2^3)C_1 + [s^3 - s^2(2p_2 + p_1) + s(p_2^2 + p_1 p_2) - p_1 p_2^2]C_{21} \\ &\quad + (s^2 - p_1 s - p_2 s + p_1 p_2)C_{22} + sC_{23} - p_1 C_{23} \\ &= (C_1 + C_{21})s^3 + [-3p_2 C_1 - (2p_2 + p_1)C_{21} + C_{22}]s^2 \\ &\quad + [3p_2^2 C_1 + (p_2^2 + 2p_1 p_2)C_{21} - (p_1 + p_2)C_{22} + C_{23}]s \\ &\qquad + (-p_2^3 C_1 - p_1 p_2^2 C_{21} + p_1 p_2 C_{22} - p_1 C_{23}). \end{aligned}$$

So if the numerator polynomial is given by the quadratic $Num(s) =$

$a_n s^2 + b_n s + c_n$, equating like terms gives

$$\begin{aligned} 0 &= C_1 + C_{21} \\ a_n &= -3p_2 C_1 - (2p_2 + p_1)C_{21} + C_{22} \\ b_n &= 3p_2^2 C_1 + (p_2^2 + 2p_1p_2)C_{21} - (p_1 + p_2)C_{22} + C_{23} \\ c_n &= -p_2^3 C_1 - p_1p_2^2 C_{21} + p_1p_2 C_{22} - p_1 C_{23}. \end{aligned} \tag{1.27}$$

The matrix-algebra approach to solving these simultaneous equations now looks like this:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -3p_2 & -(2p_2+p_1) & 1 & 0 \\ 3p_2^2 & p_2^2 + 2p_1p_2 & -(p_1+p_2) & 1 \\ -p_2^3 & -p_1p_2^2 & p_1p_2 & -p_1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_{21} \\ C_{22} \\ C_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ a_n \\ b_n \\ c_n \end{pmatrix}$$

and

$$\begin{pmatrix} C_1 \\ C_{21} \\ C_{22} \\ C_{23} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -3p_2 & -(2p_2+p_1) & 1 & 0 \\ 3p_2^2 & p_2^2 + 2p_1p_2 & -(p_1+p_2) & 1 \\ -p_2^3 & -p_1p_2^2 & p_1p_2 & -p_1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ a_n \\ b_n \\ c_n \end{pmatrix}. \tag{1.28}$$

But since C_1 and C_{23} can be found as shown above, it's also possible to find C_{21} and C_{22} using the first and third equations from the equation group (Eq. 1.27) shown above:

$$C_{21} = -C_1.$$

and Eq. 1.27 gives

$$C_{22} = \frac{c_n + (p_2^3 - p_1p_2^2)C_1 + p_1C_{23}}{p_1p_2}.$$

Using Eqs. 1.25 and 1.26 gives

$$\begin{aligned} C_1 = \left.\frac{Num(s)}{(s-p_2)^3}\right|_{s=p_1} &= \left.\frac{3s^2 - 2s + 4}{[s-(-2)]^3}\right|_{s=p_1} \\ &= \frac{3(3)^2 - 2(3) + 4}{(3+2)^3} = \frac{25}{125} = \frac{1}{5}. \end{aligned}$$

and

$$\begin{aligned} C_{23} = \left.\frac{Num(s)}{s-p_1}\right|_{s=p_2} &= \left.\frac{3s^2 - 2s + 4}{s-p_1}\right|_{s=p_2} \\ &= \frac{3(-2)^2 - 2(-2) + 4}{-2-3} = \frac{20}{-5} = -4. \end{aligned}$$

Eq. 1.27 gives

$$C_{21} = -C_1 = -\frac{1}{5}.$$

and

$$\begin{aligned} C_{22} = \frac{c_n + (p_2^3 - p_1 p_2^2)C_1 + p_1 C_{23}}{p_1 p_2} &= \frac{4 + [(-2)^3 - (3)(-2)^2]\left(\frac{1}{5}\right) + 3(-4)}{(3)(-2)} \\ &= \frac{-12}{-6} = 2. \end{aligned}$$

These values can be checked against the results of using the matrices in Eq. 1.28, which give

$$\begin{pmatrix} C_1 \\ C_{21} \\ C_{22} \\ C_{23} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -3(-2) & -[2(-2)+3] & 1 & 0 \\ 3(-2)^2 & (-2)^2 + 2(3)(-2) & -(3-2) & 1 \\ -(-2)^3 & (-3)(-2)^2 & (3)(-2) & -3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0.2 \\ -0.2 \\ 2 \\ -4 \end{pmatrix}. \quad (1.29)$$

in accordance with results shown above.

The derivative approach to finding the coefficients of the partial fractions of repeated poles also gets a bit more complicated for higher multiplicities, as you an see for the $r = 3$ case. Recall that the derivative equation is

$$C_{2n} = \frac{1}{(r-n)!} \frac{d^{r-n}}{ds^{r-n}} \left[(s - p_2)^r F(s) \right] \Big|_{s=p_2}$$

so if $r = 3$, then you can use this equation to find C_{21},C_{22}, and C_{23}. For C_{21}, second derivatives are required:

$$\begin{aligned} C_{21} &= \frac{1}{(3-1)!} \frac{d^{3-1}}{ds^{3-1}} \left[(s - p_2)^2 F(s) \right] \Big|_{s=p_2} \\ &= \frac{1}{2} \frac{d^2}{ds^2} \left[(s - p_2)^2 F(s) \right] \Big|_{s=p_2} \end{aligned}$$

while a first derivative is needed to find C_{22}

$$\begin{aligned} C_{22} &= \frac{1}{(3-2)!} \frac{d^{3-2}}{ds^{3-2}} \left[(s - p_2)^2 F(s) \right] \Big|_{s=p_2} \\ &= \frac{1}{1} \frac{d}{ds} \left[(s - p_2)^2 F(s) \right] \Big|_{s=p_2} \end{aligned}$$

and the zeroth derivative (that is, no derivative) is needed for C_{23}:

$$C_{23} = \frac{1}{(3-3)!}\frac{d^{3-3}}{ds^{3-3}}\left[(s-p_2)^2F(s)\right]\Big|_{s=p_2}$$
$$= \left[(s-p_2)^2F(s)\right]|_{s=p_2}$$

once again using $0! = 1$ and $\frac{d^0}{ds^0}[(s-p_2)^2F(s)] = (s-p_2)^2F(s)$.

For $F(s)$ given by Eq. 1.23, taking the required derivatives looks like this:

$$C_{21} = \frac{1}{2}\frac{d^2}{ds^2}\left[(s-p_2)^2F(s)\right]\Big|_{s=p_2}$$
$$= \frac{1}{2}\frac{d^2}{ds^2}\left[(3s^2-2s+4)(s-p_1)^{-1}\right]\Big|_{s=p_2}$$
$$= \frac{1}{2}\frac{d}{ds}\left[\frac{6s-2}{s-p_1} + \frac{(-1)(3s^2-2s+4)}{(s-p_1)^2}\right]\Big|_{s=p_2}$$
$$= \frac{1}{2}\left[\frac{6}{s-p_1} + \frac{(-1)(6s-2)}{(s-p_1)^2} - \frac{6s-2}{(s-p_1)^2} - \frac{(-2)(3s^2-2s+4)}{(s-p_1)^3}\right]\Big|_{s=p_2}$$

Inserting values $p_1 = 3$ and $p_2 = -2$ yields

$$C_{21} = \frac{1}{2}\left[\frac{6}{-2-3} + \frac{(-1)(6(-2)-2)}{(-2-3)^2} - \frac{6(-2)-2}{(-2-3)^2} - \frac{(-2)(3(-2)^2-2(-2)+4)}{(-2-3)^3}\right]$$
$$= \frac{1}{2}\left[\frac{6}{-5} + \frac{14}{(25} - \frac{-14}{25} - \frac{-40}{-125}\right] = \frac{1}{2}\left(\frac{-50}{125}\right) = -\frac{1}{5}.$$

The cover-up method gives C_1 as

$$C_1 = \frac{a_ns^2+b_ns+c_n}{s-p_2}\Big|_{s=p_1} = \frac{3(3)^2+(-2)(3)+4}{3-(-2)} = \frac{1}{5}$$

and

$$C_{22} = \frac{d}{ds}\left[(3s^2-2s+4)(s-p_1)^{-1}\right]\Big|_{s=p_2}$$
$$= \left[\frac{6s-2}{s-p_1} + \frac{(-1)(3s^2-2s+4)}{(s-p_1)^2}\right]\Big|_{s=p_2}$$
$$= \frac{6(-2)-2}{-2-3} + \frac{(-1)(3(-2)^2-2(-2)+4)}{(-2-3)^2}$$
$$= \frac{-14}{-5} + \frac{-20}{25} = \frac{10}{5} = 2$$

and

$$C_{23} = \left[\frac{3s^2 - 2s + 4}{s - p_1}\right]|_{s=p_2}$$

$$= \left[\frac{3(-2)^2 - 2(-2) + 4}{-2 - 3}\right] = \frac{20}{-5} = -4$$

in agreement with the results shown above.

The final section of this document contains a few on-line resources and texts that you may find helpful in understanding partial fractions.

References

Websites

https://lpsa.swarthmore.edu/BackGround/PartialFraction/PartialFraction.html

https://www.purplemath.com/modules/partfrac.htm

Texts

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