Supplemental Material for *A Student's Guide to Laplace Transforms*

Region of Convergence

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Introduction

As explained in Chapter 1 of *A Student's Guide to Laplace Transforms*, one useful characteristic of the Laplace transform is that the factor e^{-st} in the Laplace-transform integral allows the Laplace transform to exist for some time-domain functions *f(t)* for which the Fourier transform does not exist. But as shown in the examples of the Laplace transform in Chapter 2, the convergence of the Laplace-transform integral depends on the value of the generalized-frequency variable *s*, and the range of values of *s* over which the integral in the Laplace transform converges is called the Region of Convergence or ROC.

If the s-domain function *F(s)* that results from taking the Laplace transform of the time-domain function *f(t)* can be written as the ratio of two polynomials (that is, if *F(s)* is rational), the zeros of *F(s)* can be found by determining the roots of the numerator polynomial, and the poles of *F(s)* can be found by determining the roots of the denominator polynomial. None of those poles can exist within the ROC, because the value of *F(s)* is infinite at the values of *s* for which the denominator polynomial is zero, which means that the Laplace transform integral does not converge at those values of *s*. So there can be no poles within the ROC, but one or more poles can exist just outside the boundary of the ROC, as shown in the *s*-plane plots of Fig. 1.

In this figure, the shaded regions represent the ROC for three different *s*-plane functions, and each "x" represents a pole while each "o" represents a zero. As you can see, in each case the ROC is a vertical strip in the *s*-plane, extending vertically (that is, along the ω-axis) to $\pm\infty$ and with horizontal extent either from one value of σ rightward to $+\infty$ (as in Fig. 1a), leftward to $-\infty$ (as in Fig. 1b), or between two values of σ (as in Fig. 1c). Why the ROC has these characteristics and the conditions under which each of the three forms of the ROC shown in Fig. 1 occurs are explained in the following sections.

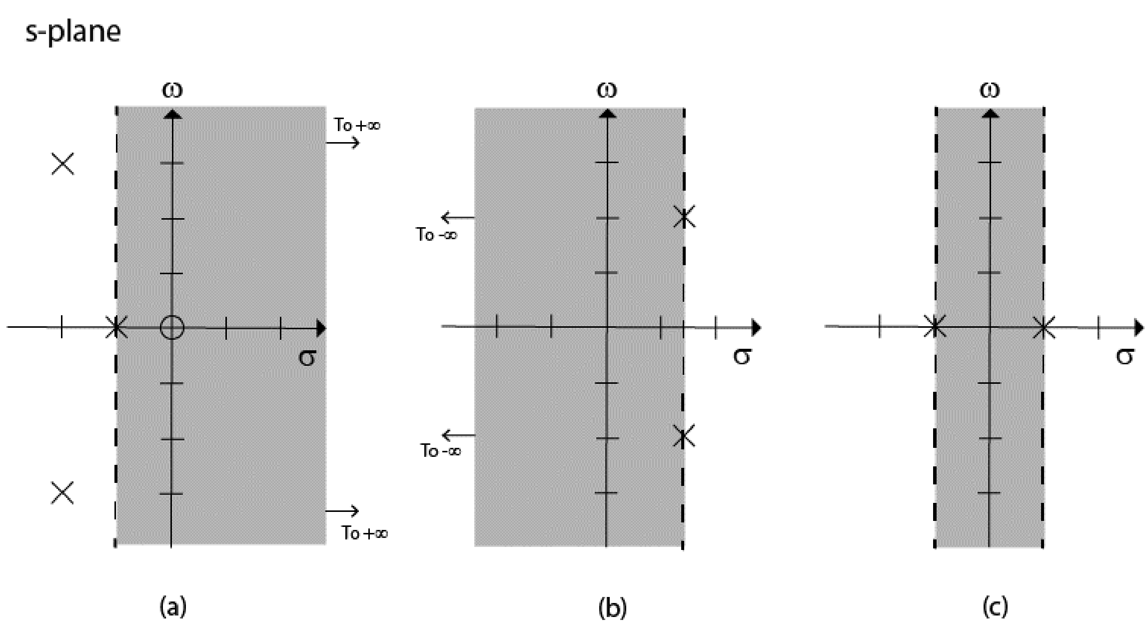


Figure 1: Three examples of Laplace-transform region of convergence (ROC).

Section 1: ROC Characteristics

To understand why the region of convergence of the Laplace transform always appears as a strip or half-plane in the *s*-plane, as shown in Fig. 1, recall the role of the real part (σ) and the imaginary part (ω) of the generalized-frequency parameter $s = \sigma + i\omega$. As described in Section 1.5 of the text, σ determines the rate of 1/e-steps in the amplitude of the product $f(t)e^{-st} = f(t)e^{-\sigma t}e^{-i\omega t}$, while ω determines the frequency of oscillation of the sine and cosine functions that make up $e^{-i\omega t}$ through Euler's relation (Eq. 1.4 of the text). Recall also that if the Laplace transform is to converge, the integration of the product $f(t)e^{-st}$ over time from 0^- to ∞ (for the one-sided transform) or from -∞ to ∞ (for the two-sided transform) must yield a finite result. So if the function $f(t)$ is itself not integrable over the relevant time, the factor $e^{-\sigma t}$ must provide sufficient amplitude reduction to make the integral of the product $f(t)e^{-\sigma t}e^{-i\omega t}$ finite. And since σ determines the rate of amplitude change of $e^{-\sigma t}$, it's reasonable to expect that the boundary of the ROC in the *s*-plane corresponds to a specific value of σ, as you can see in Fig. 1.

What does the value of the angular frequency ω contribute to the shape of the ROC? Only this: if the value of σ is sufficient to cause the integral of the product $f(t)e^{-\sigma t}e^{-i\omega t}$ to converge, then any value of ω in the factor $e^{-i\omega t}$ will not affect that convergence, since this term simply oscillates sinusoidally between the values of +1 and -1. That's why the boundary of the ROC is a vertical line in the *s*-plane, extending from $\omega = -\infty$ to $\omega = +\infty$.

Knowing that the boundary of the ROC is a vertical line at a value of σ that causes the integral of the Laplace transform to converge leaves the question of the horizontal extent of the ROC. As shown in Fig. 1, the ROC may extend rightward to $\sigma =+\infty$, or leftward to $\sigma = -\infty$, or it may be constrained to a range of values of σ. To understand the reasons for these different characteristics, it's necessary to understand the difference between time-domain functions that are right-sided, left-sided, two-sided, and finite-duration. Those differences are explained in the next section.

Section 2: Right-sided, Left-sided, Two-sided, and Finite-duration Time-domain Functions

A right-sided time-domain function has value of zero for all values of time before some starting time and extends to +∞ in the positive-time direction (usually shown as rightward on plots). Examples of right-sided functions are shown in Figure 2 a, b, and c; in these figures the three dots indicate that the function continues to infinity in the positive-t direction.

The time-domain function $f(t)$ is shown for each of these three right-sided functions is shown on the plot, and you may find it helpful to make sure you understand the effect of additive constants within the argument of the unit-step function $u(t)$. To understand that, remember that the function $u(t)$ is defined to have a value of zero whenever its argument is negative and a value of one whenever its argument is positive. So in Figure 2a with $f(t)=e^{-at}u(t)$, the $u(t)$ function steps from zero (for negative time) to one (for positive time), and that step occurs at time $t = 0$.

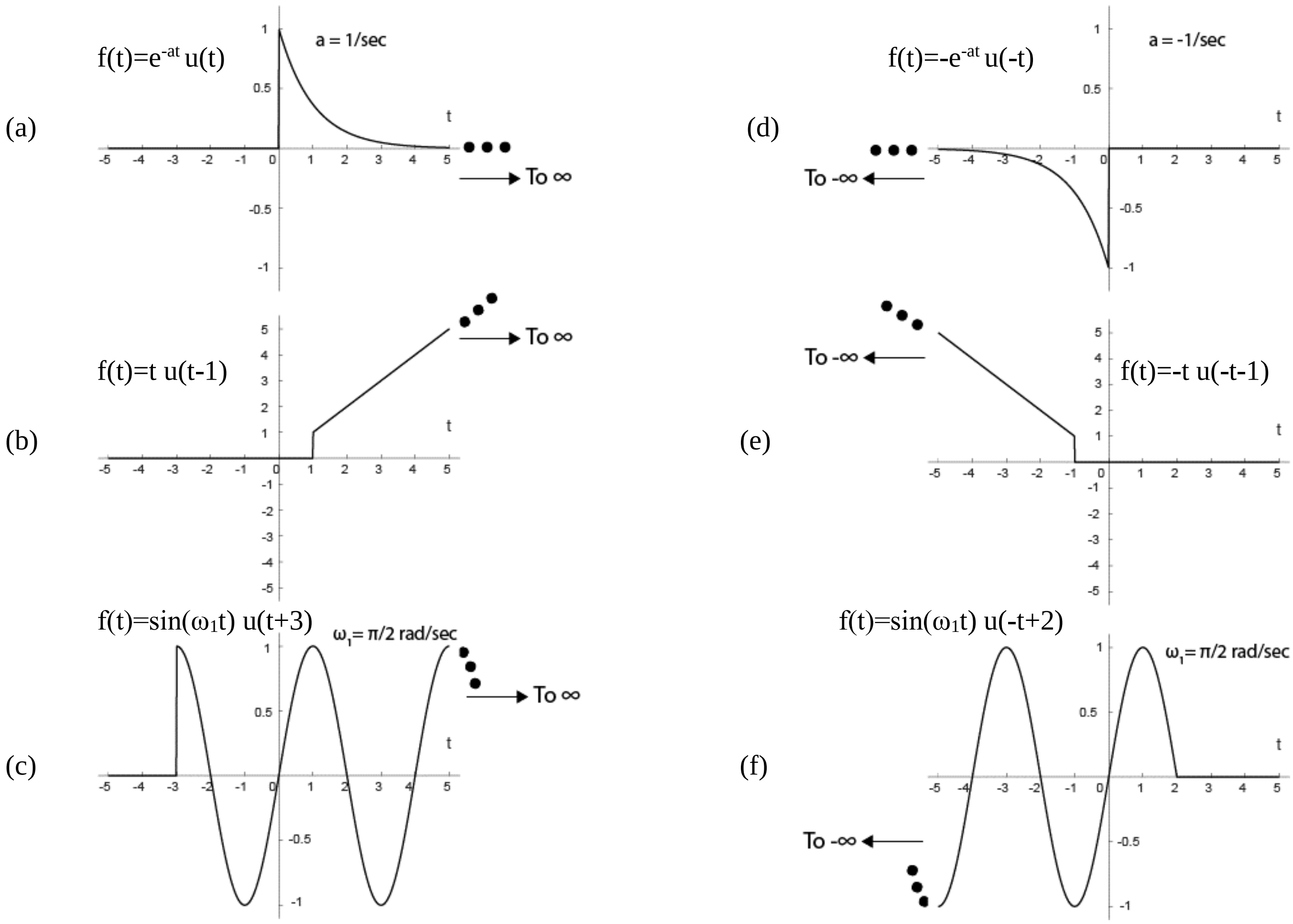


Figure 2: Examples of right-sided (a, b, and c) and left-sided (d, e, and f) functions.

Now take a look at the plot of the function $f(t)=t\ u(t-1)$ in Figure 2b. In this case, the argument (t-1) of the step function is negative whenever t is less than one and positive whenever t is greater than one. So the step from zero to one for the function $u(t-1)$ occurs at time $t = 1$. In this case, the additive constant -1 causes the function to shift one unit to the right (in the direction in which t is becoming more positive).

You can see the effect of adding a positive constant to the argument of the unit-step function in Figure 2c, in which the step from zero to one for the function $f(t)=sin(\omega_1 t)u(t+3)$ occurs at time t=-3. That's because the argument ($t+3$) of the step function is negative whenever t is less than -3 and positive whenever t is greater than -3.

Note that for right-sided time-domain functions the time at which the function steps from zero values to non-zero values can be negative, zero, or positive, as long as the non-zero portion of the function extends rightward from the starting time to positive infinity.

Once you're comfortable with the plots and equations of right-sided functions, the left-sided functions shown in Figure 2 d, e, and f should also make sense. As you can see in the figures, left-sided functions have value of zero after some ending time and non-zero values for all times less than that time, extending leftward to $t = -\infty$.

To comprehend the equations representing left-sided functions, it's important to understand not only the role of additive constants in the argument of the unit-step function, but also the effect of the minus sign in front of t in the function $u(-t)$. As you may be aware, a minus sign in front of the independent variable in a function has the effect of (horizontally) reversing the plot of that function. So in the case of the unit-step function $u(t)$, which steps from zero to one at time $t = 0$, as shown in Figure 3a, inserting a minus sign in front of the independent variable means that the function $u(-t)$ steps from one to zero at time $t = 0$, as shown in Figure 3b. That's understandable, since the unit-step function is defined as having zero value when its argument is negative, and the argument $-t$ is negative when time t is positive. Likewise, the unit-step function has value of one when its argument is positive, and the argument $-t$ is positive when time t is negative.

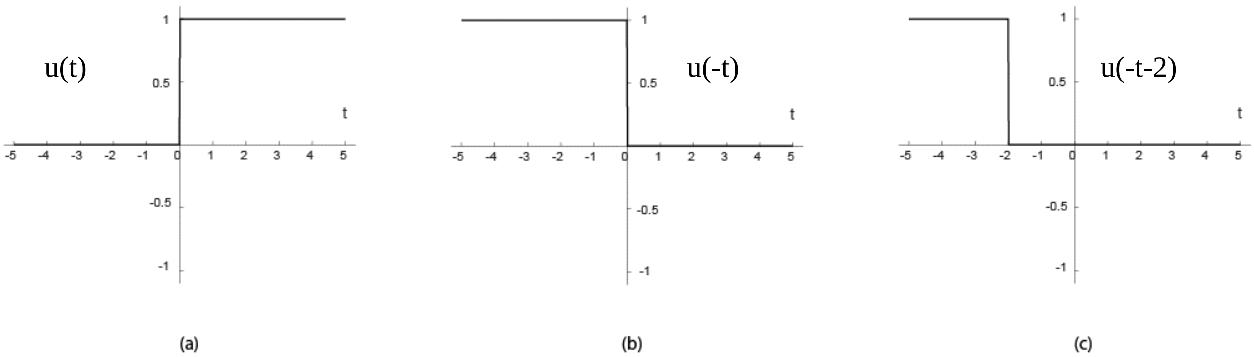


Figure 3: The time-domain unit-step functions $u(t)$ (a), $u(-t)$ (b), and $u(-t-2)$ (c).

Some students find the presence of an additive constant in addition to a minus sign in front of the argument somewhat confusing, since adding a positive constant to the argument of a function such as the reverse unit step $u(-t)$ shifts the step to the right (toward later time), while a negative additive constant in the argument of $u(-t)$ shifts the step to the left (toward earlier time), as shown in Figure 3c for the function $u(-t-2)$. The directions of these shifts can be understood by remembering that the step takes place at the time at which the argument of the unit-step function is zero, and $-t-2=0$ when $t=-2$.

Applying this logic to the left-sided functions shown in Figure 2, you should be able to understand why the non-zero values of the function $f(t)=-e^{-at}\,u(-t)$ shown in Figure 2d lie to the left of $t=0$, why the non-zero values of $f(t)=-t\,u(-t-1)$ shown in Figure 2e lie to the left of $t=-1$, and why the non-zero values of $f(t)=\sin(\omega_1 t)\,u(-t+2)$ shown in Figure 2f lie to the left of $t=+2$.

Note that for left-sided time-domain functions the time at which the function steps from non-zero values to zero values can be negative, zero, or positive, as long as the non-zero portion of the function extends leftward from the ending time to negative infinity.

The Laplace transform region of convergence for right- and left-sided functions is discussed in the next section, but before that you should consider the two-sided and finite-duration functions shown in Figure 4. As you can see in the a,b, and c portions of this figure, two-sided time-domain functions extend from $t=-\infty$ to $t=+\infty$, while the finite-duration functions shown in the c, d, and e portions of the figure have both a starting time and an ending time and do not extend to infinity in either direction.

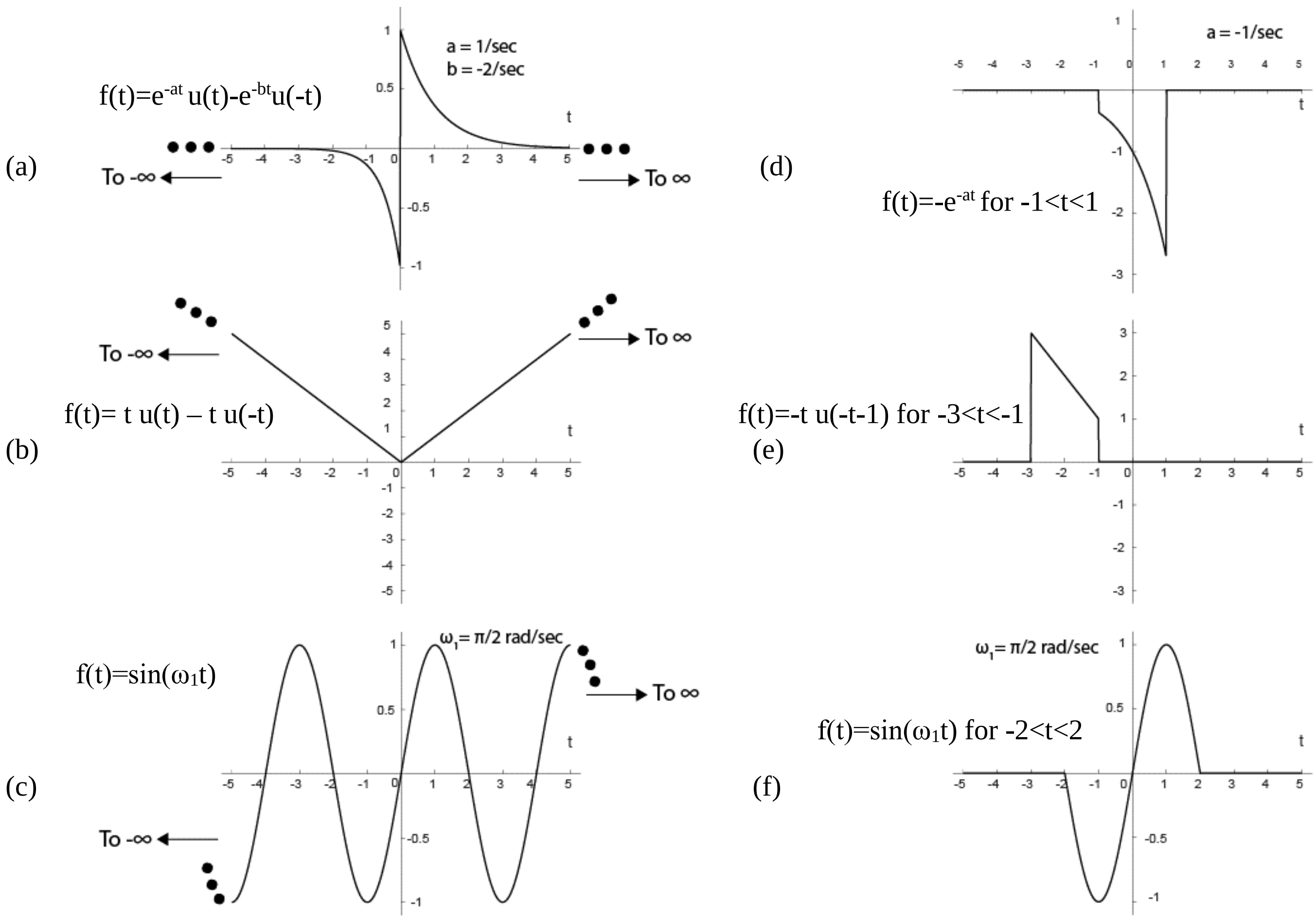


Figure 4: Examples of two-sided (a, b, and c) and finite-duration (d, e, and f) functions.

Note also that two-sided functions can be written as the sum of a left-sided function and a right-sided functions using the unit-step function *u(t)* (possibly offset by a constant) for the right-sided portion and the reverse unit-step function *u(-t)* (also possibly offset) for the left-sided portion of the composite function.

With these definitions, plots, and equations of right-sided, left-sided, two-sided, and finite-duration waveforms in hand, you should be ready to understand the shape of the ROC for each of these types of time-domain function. That's the subject of the next section.

Section 3: ROC for Right-sided Functions

The reason for the somewhat lengthy discussion of right-sided and left-sided functions in the previous section is that there is a significant difference in the region of convergence of the Laplace transform for time-domain functions that extend rightward to positive infinity and those that extend leftward to negative infinity.

If you've worked through the examples in Chapter 2 of the text, you may have noticed that in each of those examples the ROC extends from a specified value of *s* to infinity, such as *s*>0 for constant time-domain functions $f(t)=c$ and *s*>a for exponential functions $f(t)=e^{at}$. That's because the Laplace transform used in the examples in Chapter 2 is the unilateral (one-sided) transform, and the lower limit of $t=0^-$ of the integral in that transform ensures that only the positive-time values of $f(t)$ contribute to the result $F(s)$ of the transformation process. Hence the unilateral Laplace transform provides the generalized-frequency spectrum of causal functions (that is, functions with zero value for all negative time), and causal functions are right-sided by definition.

To understand why the ROC for right-sided functions extends rightward in the *s*-domain from some value of σ to positive infinity, consider the right-sided time-domain exponential function $f(t)=e^{-3t}u(t)$. Plots of the product of this function with the factor $e^{-\sigma t}$ from the Laplace-transform integral are shown in Figure 5a for values of σ ranging from +1 to -4.

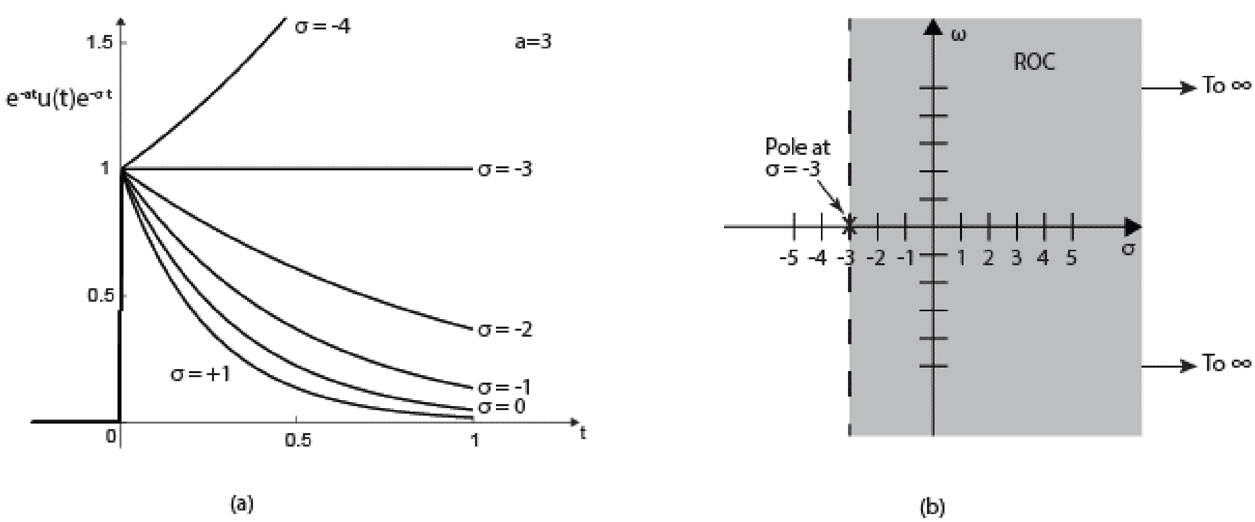


Figure 5: (a) Time-domain plots of $e^{-3t}u(t)\ e^{-\sigma t}$ and (b) ROC for Laplace transform of $f(t)= e^{-3t}u(t)$.

Look first at the curve for σ=+1. Since both factors e^{-3t} and $e^{-\sigma t}$ are decreasing exponentials in this case, their product, which is the integrand of the Laplace-transform integral, is also a decreasing exponential, which means the Laplace transform converges. So σ=+1 is clearly within the ROC, as is any larger positive value of σ, which would cause the $e^{-\sigma t}$ factor to decrease even more rapidly. So the ROC extends infinitely far in the positive-σ direction, as shown in Figure 5b.

Now consider what happens if σ is negative. In those cases, the exponent of the factor $e^{-\sigma t}$ is positive for positive time, which means the factor $e^{-\sigma t}$ is an increasing exponential in that case. But as you can see in Figure 5a, the curves of the product $e^{-3t}u(t)e^{-\sigma t}$ are still decreasing exponentials for σ=-1 and σ=-2, since the decrease in the e^{-3t} factor dominates the increase in the $e^{-\sigma t}$ factor. That wouldn't be true for time-domain functions that don't decrease with increasing time as rapidly as the function $f(t)=e^{-3t}u(t)$, which is why the ROC for many functions does not extend into the negative-σ range. But even for right-sided time-domain functions that increase over time, the $e^{-\sigma t}$ factor can be made to reduce the integrand of the Laplace transform sufficiently for the integral to converge, provided the function $f(t)$ is of exponential order, as described in Chapter 1 of the text.

Of course, as σ becomes increasingly negative, the increasing exponential factor $e^{-\sigma t}$ will offset or exceed the decreasing factor e^{-3t}. You can see that happening for the curve of σ=-3 in Figure 5a, in which case the product $e^{-3t}u(t)e^{-\sigma t}$ remains constant as time increases, which means the Laplace integral over all positive time will not converge for that value of σ. That's why the dashed line signifying the lower bound of the ROC is shown at σ=-3 in Figure 5b. Note that additional poles may exist (the number of poles depends on the time-domain function), but no pole can exist within the ROC, so those poles must lie to the left of the ROC for a right-sided function. Hence it's the rightmost pole that sets the boundary of a right-sided function.

And for σ=-4, the product $e^{-3t}u(t)e^{-\sigma t}$ is an increasing exponential, which means that this (and all more negative values of σ) are clearly outside the ROC for the Laplace transform of $f(t)=e^{-3t}u(t)$.

Note that this logic is reversed when $t<0$ because the factors e^{-3t} and $e^{-\sigma t}$ are both increasing exponentials when time is negative (and becoming more negative) and σ is positive. Doesn't that ruin the convergence for sufficiently large positive values of σ? No, because for right-sided functions, $f(t)$ has value of zero at times to the left of the start time, and those zero values prevent the product $e^{-3t}u(t)e^{-\sigma t}$ from growing without limit as t approaches negative infinity.

Section 4: ROC for Left-sided Functions

A modified version of the analysis in the previous section of the ROC for right-sided functions can be applied to left-sided functions, as shown in Figure 6. The time-domain function used as an example in this case is $f(t)=e^{2t}u(-t)$, which has non-zero value only when time is negative, and which decreases exponentially from a value of one at time $t=0^-$ (just before time $t=0$) toward zero as time approaches negative infinity. The product of the two terms $e^{2t}u(-t)$ and $e^{-\sigma t}$ (that is, the integrand of the Laplace transform) for several values of σ is shown in the (a) portion of Figure 6, and the ROC for this left-sided sequence is shown in the (b) portion of the figure. As this example demonstrates, the ROC for left-sided functions extends leftward to -∞.

The reason for this can be understood by looking at the curves representing the product $e^{2t}u(-t)e^{-\sigma t}$ in Figure 6a. For the curve representing σ=-1, both factors e^{2t} and $e^{-\sigma t}$ are decreasing exponentials as time becomes more negative, which means that their product is also a decreasing exponential, so the Laplace transform converges for this value of σ. Thus σ=-1 is within the ROC, as is any larger negative value of σ, which would cause the $e^{-\sigma t}$ factor to decrease even more rapidly. So for this left-sided function, the ROC extends infinitely far in the negative-σ direction.

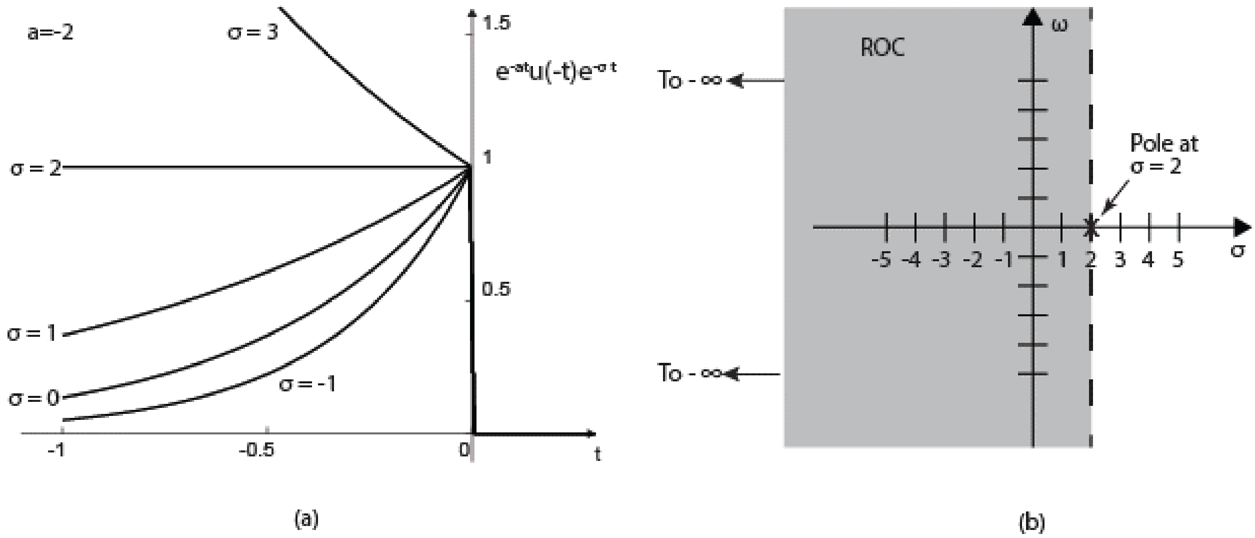


Figure 6: (a) Time-domain plots of $e^{2t}u(-t)\ e^{-\sigma t}$ and (b) ROC for Laplace transform of $f(t)= e^{2t}u(-t)$.

And if σ is positive, the exponent of the factor $e^{-\sigma t}$ is positive for negative time, which means the factor $e^{-\sigma t}$ is an increasing exponential. But the curve of the product $e^{2t}u(t)e^{-\sigma t}$ is a decreasing exponential for σ=+1 because the decrease in the e^{2t} factor dominates the increase in the $e^{-\sigma t}$ factor, so σ=+1 is within the ROC in this case. However, large positive values of σ cause the increasing exponential factor $e^{-\sigma t}$ to offset or exceed the decreasing factor e^{2t}, as you can see for the curves of σ=+2 and +3 in Figure 6a. That means the Laplace integral will not converge for those values of σ, and that's why the dashed line signifying the lower bound of the ROC is shown at σ=+2 in Figure 6b. As in the case of right-sided functions, additional poles may exist, but in this case those poles must lie to the right of the ROC, so it's the leftmost pole that sets the boundary of a left-sided function.

Section 5: ROC for Two-sided and Finite-Duration Functions

Recall from the discussion in Section 2 that two-sided time-domain functions extend from t=-∞ to t=+∞, as shown on the left side of Figure 4. The ROC for such functions can be understood by considering the two-sided function to be the combination of a left-sided function, extending from t=-∞ to an arbitrary point in time, and a right-sided function, extending from that same point in time to t=+∞. Since the Laplace transform is a linear process, the transform of the sum of two functions is just the sum of the transforms of those functions, which means that the Laplace transforms of both the left-sided function and the right-sided function must converge.

But note that for the Laplace transform of the combined function to converge, the Laplace transforms of the left-sided and right-sided functions must not only converge, but they must also converge over some range of the same values of σ. Consider, for example, the two sided time-domain function $f(t)=e^{-3t}\,u(t)-e^{2t}u(-t)$, for which the product $[e^{-3t}\,u(t)-e^{2t}u(-t)]e^{-\sigma t}$ is shown in Figure 7a. As discussed in Sections 3 and 4 above, the ROC of the Laplace transform for the right-sided portion of this function extends rightward from σ=-3 to σ=+∞, and the ROC for the transform of the left-sided portion extends leftward from σ=2 to σ=-∞. Since the Laplace transform of the combined function will not converge at any value of σ at which the transform of either of the two constituent functions does not converge, the ROC for the combined function must be the overlap of the regions of convergence of the constituent functions. You can see that overlap in Figure 7b, which in this case extends from σ=-3 to σ=+2. This is an example of the characteristic that the ROC of two-sided functions must be bounded by poles on each side.

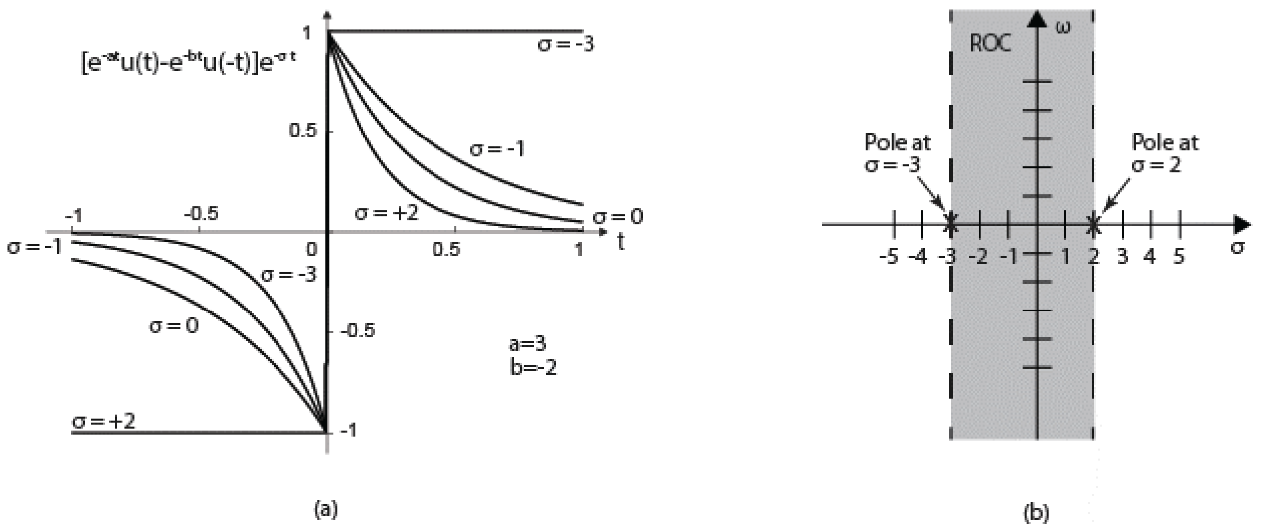


Figure 7: (a) Time-domain plots of $[e^{-3t}u(t)- e^{2t}u(-t)]e^{-\sigma t}$ and (b) ROC for Laplace transform of the two-sided function $f(t)= e^{-3t}u(t)$.

Of course, it's possible to construct composite two-sided functions for which no overlap exists between the ROC of the left-sided function and the ROC of the right-sided function. One example of that is the function $f(t)=e^{3t}\,u(t)-e^{-2t}u(-t)$, for which the ROC of the right-sided function $e^{3t}\,u(t)$ extends from σ=+3 to σ=+∞, while ROC of the left-sided function $e^{-2t}\,u(t)$ extends from σ=-2 to σ=-∞. Since there's no overlap between these regions of convergence (that is, no value of σ at which both the transforms of the left-sided and the right-sided functions converge), the Laplace transform of the combined function has no ROC.

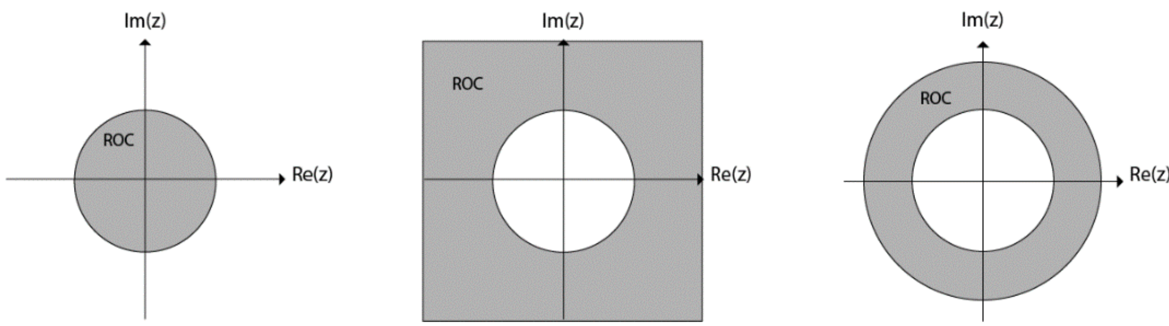
The ROC for the Laplace transform of a finite-duration time-domain function is quite straightforward. If the time-domain function doesn't extend to infinity in either direction, then the product $f(t)e^{-\sigma t}$ in the Laplace transform integral will have value of zero outside the time range over which the finite-duration function $f(t)$ exists. So as long as $f(t)$ is itself absolutely integrable, multiplication by the factor $e^{-\sigma t}$ will not prevent the Laplace integral from converging (the multiplying factor can cause the product $f(t)e^{-\sigma t}$ to grow, but not to blow up to infinitely large value if $f(t)$ goes to zero at some time). That means that the ROC covers the entire *s*-plane for such finite-duration functions.

One final note about the ROC of the Laplace transform: it's possible for a right-sided sequence and a left-sided sequence to have the same Laplace transform, such as $e^{at}\,u(t)$ and $-e^{at}\,u(-t)$ which both have the Laplace transform $F(s)=1/(s-a)$. In such cases, determining $f(t)$ from $F(s)$ can only be done without ambiguity if the ROC is specified in addition to $F(s)$. This is not an issue for the unilateral (one-sided) Laplace transform and causal functions, which is why the ROC is usually not emphasized and may be omitted entirely in some texts. But for the bilateral (two-sided) Laplace transform, it's essential to know the ROC when attempting to determine the time-domain function *f(t)* for a given s-domain function *F(s)*.

Section 6: Z-Transform Region of Convergence

If you've worked through the discussion of the relationship between the Laplace-transform s-plane and the Z-transform z-plane, the characteristics of the ROC for the Z-transform of left-sided, right-sided, and two-sided time-domain sequences should come as no surprise.

Remember that lines of constant σ (vertical lines in the s-plane) correspond to circles in the z-plane, with the ω-axis (where σ=0) in the s-plane mapping to the unit circle in the z-plane. That means that the left half-plane of the s-plane maps into the interior of the unit circle in the z-plane, while the right half-plane of the s-plane maps onto the exterior of the unit circle of the z-plane.



(a) (b) (c)

Figure 8: Examples of Z-transform ROC for (a) left-, (b) right-, and (c) two-sided sequences.

And since the ROC of the Laplace transform of a left-sided function extends leftward from some value of σ to negative infinity in the s-plane, the ROC of the Z-transform of a left-sided sequence is the interior of a circle in the z-plane, as shown in Figure 8a. The radius of the circle is determined by the value of the pole closest to the origin (z=0).

Similarly, since the ROC of the Laplace transform of a right-sided function extends rightward from some value of σ to positive infinity in the s-plane, the ROC of the Z-transform of a right-sided sequence is the exterior of a circle in the z-plane, as shown in Figure 8b. In this case the radius of the circle is determined by the value of the pole farthest from the origin.

As described in the previous section, the ROC for the Laplace transform of two-sided time-domain functions is a strip in the s-plane, corresponding to an annulus in the z-plane, as shown in Figure 8c. The annulus is bounded by poles at its inner and outer edges.