Chapter 6

Exercise Solutions

1. From Eq. 6.1, density = mass / volume

   So, mass = density × volume

   styrofoam: density = 75 kg/m^3

   volume = 1 cm^3

   A unit conversion factor will be needed in order to cancel m and cm. Since m and cm are cubed, the factor will have to be cubed as well:

$$\begin{aligned}\text{mass} = \text{density} \times \text{volume} &= \left(75\frac{\text{kg}}{\cancel{\text{m}^3}}\right) \times (1\,\cancel{\text{cm}^3}) \times \left(\frac{1\,\cancel{\text{m}}}{100\,\cancel{\text{cm}}}\right)^3 \\ &= \frac{75}{10^6}\text{ kg} = 7.5 \times 10^{-5}\text{ kg, or 0.75 g (ans.)}\end{aligned}$$

   rock: density = 3,000 kg/m^3

   volume = 1 cm^3

$$\begin{aligned}\text{mass} = \text{density} \times \text{volume} &= \left(3000\frac{\text{kg}}{\cancel{\text{m}^3}}\right) \times (1\,\cancel{\text{cm}^3}) \times \left(\frac{1\,\cancel{\text{m}}}{100\,\cancel{\text{cm}}}\right)^3 \\ &= \frac{3000}{10^6}\text{ kg} = 3 \times 10^{-3}\text{ kg, or 3 g (ans.)}\end{aligned}$$

2. mass of car: remains the same when compacted, because no material is being added or lost (just compacted).

   volume of car: decreases when compacted because it is being crushed to fit in a smaller space (which is the whole purpose of compacting).

   average density of car: increases when compacted because the empty/air cavities within it are being filled in with metal and glass. Also, mathematically, density = mass / volume increases when volume decreases.

3. From Eq. 6.1, density = mass / volume

   volume of a sphere is: $V = \frac{4}{3}\pi R^3$

   Using the absolute method:

   $R_{\text{Jupiter}} = 71,500$ km ; $R_{\text{Mercury}} = 2,440$ km

   $m_{\text{Jupiter}} = 1.9 \times 10^{27}$ kg ; $m_{\text{Mercury}} = 3.3 \times 10^{23}$ kg

$$\text{density}_\text{J} = \frac{m_\text{J}}{V_\text{J}} = \frac{m_\text{J}}{\frac{4}{3}\pi R_\text{J}^3} = \frac{1.9 \times 10^{27}\text{ kg}}{\frac{4}{3}\pi(71,500\text{ km})^3} = 1.24 \times 10^{12}\text{ kg/km}^3$$

$$\text{density}_\text{M} = \frac{m_\text{M}}{V_\text{M}} = \frac{m_\text{M}}{\frac{4}{3}\pi R_\text{M}^3} = \frac{3.3 \times 10^{23}\text{ kg}}{\frac{4}{3}\pi(2,440\text{ km})^3} = 5.42 \times 10^{12}\text{ kg/km}^3$$

   So the density of Mercury is higher. To see how many times more,

$$\frac{\text{density}_\text{M}}{\text{density}_\text{J}} = \frac{5.42 \times 10^{12}\text{ kg/km}^3}{1.24 \times 10^{12}\text{ kg/km}^3} = 4.37\text{ (ans.)}$$

   So the density of Mercury is 4.37 times greater than Jupiter.

   OR, using ratios (which is a good approach for "compare" problems like this one),

$$\frac{\text{density}_\text{M}}{\text{density}_\text{J}} = \frac{\frac{m_\text{M}}{\cancel{\frac{4}{3}\pi} R_\text{M}^3}}{\frac{m_\text{J}}{\cancel{\frac{4}{3}\pi} R_\text{J}^3}} = \frac{m_\text{M}}{R_\text{M}^3} \times \frac{R_\text{J}^3}{m_\text{J}} = \frac{m_\text{M}}{m_\text{J}} \times \left(\frac{R_\text{J}}{R_\text{M}}\right)^3$$

$$= \left(\frac{3.3 \times 10^{23}\ \cancel{\text{kg}}}{1.9 \times 10^{27}\ \cancel{\text{kg}}}\right) \times \left(\frac{71,500\ \cancel{\text{km}}}{2,440\ \cancel{\text{km}}}\right)^3 = 4.37\text{ (ans.)}$$

   This is the same answer as above.

4. From Eq. 6.5, $v_{\text{esc}} = \sqrt{\frac{2Gm}{R}}$

$R_{\text{surface}} = 6371$ km

$R_{\text{orbit}} = 6371$ km + 350 km = 6721 km

$$\frac{v_{\text{esc, orbit}}}{v_{\text{esc, surface}}} = \frac{\sqrt{\frac{2Gm_{\text{Earth}}}{R_{\text{orbit}}}}}{\sqrt{\frac{2Gm_{\text{Earth}}}{R_{\text{surface}}}}} = \sqrt{\frac{\cancel{2Gm}_{\text{Earth}}}{R_{\text{orbit}}} \times \frac{R_{\text{surface}}}{\cancel{2Gm}_{\text{Earth}}}}$$
$$= \sqrt{\frac{R_{\text{surface}}}{R_{\text{orbit}}}} = \sqrt{\frac{6371\ \cancel{\text{km}}}{6721\ \cancel{\text{km}}}} = 0.97 \text{ (ans.)}$$

The escape velocity from low Earth orbit is 97% of the escape velocity from the surface – or almost the same.

5. From Eq. 6.5, $v_{\text{esc}} = \sqrt{\frac{2Gm}{R}}$

$m = m_{\text{Sun}} = 1.99 \times 10^{30}$ kg

R = radius of Earth orbit = 1 AU = 1.50×10^{11} m

$$v_{\text{esc}} = \sqrt{\frac{2(6.67 \times 10^{-11}\,\frac{\text{m}^{\cancel{3}2}}{\cancel{\text{kg}}\,\text{s}^2})(1.99 \times 10^{30}\ \cancel{\text{kg}})}{1.50 \times 10^{11}\ \cancel{\text{m}}}} = 4.21 \times 10^4 \text{ m/s, or } 42.1 \text{ km/s (ans.)}$$

6. From Eq. 6.5, $v_{\text{esc}} = \sqrt{\frac{2Gm}{R}}$

$m = m_{\text{Sun}} = 1.99 \times 10^{30}$ kg

R = radius of Earth orbit = 1 AU = 1.50×10^{11} m

$$v_{\text{esc}} = \sqrt{\frac{2(6.67 \times 10^{-11}\,\frac{\text{m}^{\cancel{3}2}}{\cancel{\text{kg}}\,\text{s}^2})(1.99 \times 10^{30}\ \cancel{\text{kg}})}{1.50 \times 10^{11}\ \cancel{\text{m}}}} = 4.21 \times 10^4 \text{ m/s, or } 42.1 \text{ km/s (ans.)}$$

The answer is the same as the previous exercise, because the mass of the Sun has not changed (m) and the distance from it (R) hasn't changed either.

7. From Eq. 6.6, $R_s = \frac{2Gm}{c^2}$

   a) $m = 1$ kg

   $$R_s = \frac{2\left(6.67\times10^{-11}\frac{\text{m}^3}{\text{kg s}^2}\right)(1\ \text{kg})}{(3\times10^8\ \text{m/s})^2} = 1.48\times10^{-27}\ \text{m (ans.)}$$

   Which is much much smaller than even a single atom.

   b) $m = 1.99\times10^{30}$ kg $(= 1\ \text{M}_\odot)$

   $$\frac{R_{\text{s, Sun}}}{R_{\text{s, 1kg}}} = \frac{2Gm_{\text{Sun}}/c^2}{2Gm_{\text{1kg}}/c^2} = \frac{m_{\text{Sun}}}{m_{\text{1kg}}} = \frac{1.99\times10^{30}\ \text{kg}}{1\ \text{kg}}$$

   $$R_{\text{s, Sun}} = 1.99\times10^{30}R_{\text{s, 1kg}} = (1.99\times10^{30})(1.48\times10^{-27}\ \text{m})$$
   $$= 2950\ \text{m, or } 2.95\ \text{km (ans.)}$$

   c) $m_{\text{big}} = 4\times10^6\ \text{M}_\odot$ (this is four million times the mass in the previous part, which calls for the ratio method)

   $$\frac{R_{\text{s, big}}}{R_{\text{s, Sun}}} = \frac{2Gm_{\text{big}}/c^2}{2Gm_{\text{Sun}}/c^2} = \frac{m_{\text{big}}}{m_{\text{Sun}}} = \frac{4\times10^6\ \text{M}_\odot}{1\ \text{M}_\odot} = 4\times10^6$$

   $$R_{\text{s, big}} = 4\times10^6R_{\text{s, Sun}} = (4\times10^6)(2.95\ \text{km}) = 11.8\times10^6\ \text{km (ans.)}$$

8. From Eq. 6.5, $v_{\text{esc}} = \sqrt{\frac{2Gm}{R}}$

   From Eq. 6.6, $R_s = \frac{2Gm}{c^2}$

   For $m = 1\ \text{M}_\odot$ and $R = R_s$:

   $$v_{\text{esc}} = \sqrt{\frac{2G(1\ \text{M}_\odot)}{\frac{2G(1\ \text{M}_\odot)}{c^2}}} = \sqrt{2G(1\ \text{M}_\odot)\times\frac{c^2}{2G(1\ \text{M}_\odot)}} = \sqrt{c^2} = c$$

   For $m = 10\ \text{M}_\odot$ and $R = R_s$:

   $$v_{\text{esc}} = \sqrt{\frac{2G(10\ \text{M}_\odot)}{\frac{2G(10\ \text{M}_\odot)}{c^2}}} = \sqrt{2G(10\ \text{M}_\odot)\times\frac{c^2}{2G(10\ \text{M}_\odot)}} = \sqrt{c^2} = c$$

   v_{esc} at the event horizon is always c by definition, regardless of mass

9. From Eq. 6.5, $v_{\mathrm{esc}} = \sqrt{\frac{2Gm}{R}}$

$v_{\mathrm{esc,\,4R}_s} = \sqrt{\frac{2Gm}{4R_s}}$ Escape speed at a distance of $R = 4\mathrm{R_s}$

From Eq. 6.6, $R_{\mathrm{s}} = \frac{2Gm}{c^2}$

$$v_{\mathrm{esc,\,4R}_s} = \sqrt{\frac{2Gm}{4\left(\frac{2Gm}{c^2}\right)}} = \sqrt{\cancel{2Gm} \times \frac{c^2}{4(\cancel{2Gm})}} = \sqrt{\frac{c^2}{4}} = \frac{c}{2}$$

Mass canceled. So at $R = 4R_{\mathrm{s}}$, escape speed $= c/2$, regardless of mass.

So for a 1 $\mathrm{M}_\odot$ black hole, v_{esc} at $R = 4R_{\mathrm{s}}$ ($c/2$) is half of v_{esc} at $R = R_{\mathrm{s}}$ (c).

And for a 10 $\mathrm{M}_\odot$ black hole, v_{esc} at $R = 4R_{\mathrm{s}}$ ($c/2$) is half of v_{esc} at $R = R_{\mathrm{s}}$ (c).

The same is true for a black hole of any mass.

10. For distances doubling:

$d_{\mathrm{initial}} = \frac{d}{2}$

$d_{\mathrm{final}} = d$ (because distance doubled)

$$\text{speed} = \frac{\text{distance covered}}{\text{time}} = \frac{d - \frac{d}{2}}{t} = \frac{\frac{d}{2}}{t} = \frac{d}{2t}$$

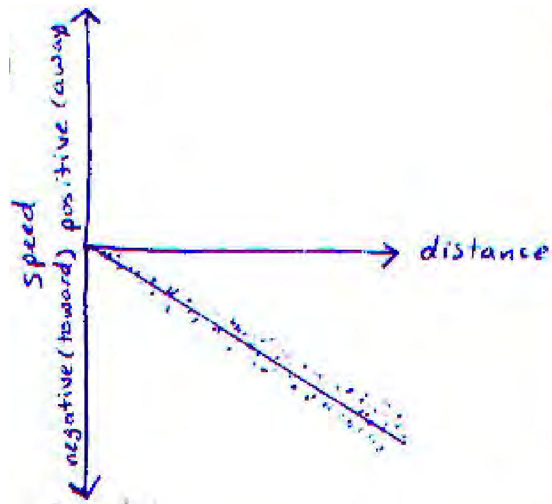
This is half the speed of galaxy 1, and one-quarter the speed of galaxy 2 (d/t and $2d/t$, respectively) over the same time interval.

For distances tripling:

galaxy 2: goes from distance $2d$ to $6d$, covering a total distance of $6d - 2d = 4d$. Doing so over time t, its speed is $v = 4d/t$.

new galaxy: goes from distance $d/2$ to $3d/2$, covering a total distance of $3d/2 - d/2 = d$. Doing so over time t, its speed is $v = d/t$. Again, this is one-quarter the speed of galaxy 2. This makes sense because it is also one-quarter the distance of galaxy 2.

11. The slope would be negative as shown in the figure below, and speeds would all be negative, because all galaxies would be moving toward us rather than away.



More distant galaxies would be moving faster toward us (more negative speeds) if their speeds were proportional to their distances. Very nearby galaxies would be moving toward us only very slowly. If galaxies all continued to move according to this pattern, all galaxies would collide with ours at the same time: some time in the finite future. (Like a reverse Big Bang, or "Big Crunch".)

12. two points: ($x_1 =$ 500 Mpc, $y_1 = 35,000$ km/s)

($x_2 = 4,000$ Mpc, $y_2 = 280,000$ km/s)

$$H_0 = \text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{280,000 \text{ km/s} - 35,000 \text{ km/s}}{4,000 \text{ Mpc} - 500 \text{ Mpc}}$$

$$= \frac{245,000 \text{ km/s}}{3,500 \text{ Mpc}} = 70 \, \frac{\text{km/s}}{\text{Mpc}} \text{ (ans.)}$$

13. two points: $(x_1 = \quad 0 \text{ Mpc}, \quad y_1 \ = \quad 0 \text{ km/s})$

$$(x_2 = 4,000 \text{ Mpc}, y_2 = 280,000 \text{ km/s})$$

$$H_0 = \text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{280,000 \text{ km/s} - 0 \text{ km/s}}{4,000 \text{ Mpc} - 0 \text{ Mpc}}$$

$$= \frac{280,\cancel{000} \text{ km/s}}{4,\cancel{000} \text{ Mpc}} = \frac{280 \text{ km/s}}{4 \text{ Mpc}} = 70 \frac{\text{km/s}}{\text{Mpc}} \text{ (ans.)}$$

This is the same answer as the previous question.

14. Reversing the order of subtraction:

$$H_0 = \text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{0 \text{ km/s} - 280,000 \text{ km/s}}{0 \text{ Mpc} - 4,000 \text{ Mpc}}$$

$$= \frac{-280,\cancel{000} \text{ km/s}}{-4,\cancel{000} \text{ Mpc}} = \frac{-280 \text{ km/s}}{-4 \text{ Mpc}} = 70 \frac{\text{km/s}}{\text{Mpc}} \text{ (ans.)}$$

This is still the same answer as the previous two questions. No matter which two points you choose on a straight line, and no matter which order you subtract them, you'll get the same slope.

15. From Eq. 6.9, $d = v/H_0$

$v = 15,000$ km/s

$H_0 = 70 \frac{\text{km/s}}{\text{Mpc}}$

$$d = \frac{15,000 \cancel{\text{km/s}}}{70 \frac{\cancel{\text{km/s}}}{\text{Mpc}}} = 214 \text{ Mpc (ans.)}$$

16. From Eq. 6.10, $T_0 = \frac{1}{H_0}$

$H_0 = 69 \frac{\text{km/s}}{\text{Mpc}}$

$$T_0 = \frac{1}{69 \frac{\text{km/s}}{\text{Mpc}}} = \frac{1 \text{ Mpc}}{69 \text{ km/s}} = \frac{10^6 \cancel{\text{pc}}}{69 \cancel{\text{km}}/\text{s}} \times \left(\frac{3.09 \times 10^{13} \cancel{\text{km}}}{1 \cancel{\text{pc}}}\right)$$

$$= 4.48 \times 10^{17} \cancel{\text{s}} \times \left(\frac{1 \text{ yr}}{3.16 \times 10^7 \cancel{\text{s}}}\right) = 14.2 \times 10^9 \text{ yr (ans.)}$$

Or, using ratios

$$\frac{T_{0,69}}{t_{0,70}} = \frac{1/H_{0,69}}{1/H_{0,70}} = \frac{H_{0,70}}{H_{0,69}} = \frac{70\ \cancel{\frac{\text{km/s}}{\text{Mpc}}}}{69\ \cancel{\frac{\text{km/s}}{\text{Mpc}}}} = \frac{70}{69} = 1.014$$

$$T_{0,69} = 1.014\, T_{0,70} = 1.014 \times (14.0 \times 10^9\ \text{yr}) = 14.2 \times 10^9\ \text{yr}$$

For $H_0 = 75\ \frac{\text{km/s}}{\text{Mpc}}$, and again using ratios:

$$\frac{T_{0,75}}{t_{0,70}} = \frac{1/H_{0,75}}{1/H_{0,70}} = \frac{H_{0,70}}{H_{0,75}} = \frac{70\ \cancel{\frac{\text{km/s}}{\text{Mpc}}}}{75\ \cancel{\frac{\text{km/s}}{\text{Mpc}}}} = \frac{70}{75} = 0.933$$

$$T_{0,75} = 0.933\, T_{0,70} = 0.933 \times (14.0 \times 10^9\ \text{yr}) = 13.06 \times 10^9\ \text{yr}$$

It makes sense that the age of the Universe should be shorter if the Hubble constant is larger, because they are inversely proportional. If the Universe had been expanding faster (larger Hubble constant), then it would have taken it less time (shorter age) to get to its present size.

17. Initially the object is at our location (zero distance away) and moving very rapidly away. It continues moving away, but at a slower and slower ate. Then, halfway through the total time interval ($t_{\text{end}}/2$) it comes to rest at its maximum distance away. Then it gradually starts to move back toward the beginning point, slowly at first, then faster and faster: Like a ball on an elastic band that is fired away and snaps back.

18. Only curve (1) shows a recollapsing Universe, because it alone shows "size" returning toward zero at a time in the future. It is the only curve that intersects the x-axis (y=0) again after the big bang.

19. youngest (1), (2), (3), (4) oldest

20. (2) The slope was steeply upward at first, just after the Big Bang, so the Universe was expanding quickly at first. But immediately and gradually,

the expansion began to slow, so the slope grows shallower. But the slope is ever positive, so the Universe is ever expanding, albeit ever slowing in its expansion. The slope of the curve asymptotically approaches zero slope (flat), so the Universe will approach a cessation of its expansion as $t \to \infty$. Thus the Universe will approach a maximum "size", but will never quite reach it and will never reverse or recollapse.

(4) The very steep initial slope shows that the Universe expanded very rapidly at first. The quickly flattening slope (concave down, yet still going upward) shows that the outward expansion continued, but at a slowing rate. Then gradually, the slope becomes concave upward again, becoming steeper rather than flatter again, which shows that the expansion began to accelerate, or speed up. That acceleration continues to this day, and into the future. Note that the slope was/is never negative, always positive, so the expansion was never a recollapsing motion – always an outward expansion.

21. An infinite rate of expansion corresponds to an infinite (vertical) slope on the graph. Only case (4).

22. Scenario (4) is the only one that shows an accelerating, increasing expansion rate.