Center of Mass

The purpose of this document is to explain why the concept of the center of mass is useful, provide several different ways to visualize the center of mass, and to show the mathematics of how to calculate the center of mass for systems consisting of a finite number of objects. You'll also find several links to helpful websites that offer a more sophisticated mathematical treatment.

The center of mass (CoM) is the "average position" of mass within an object or a system of objects. The concept of average position of mass is explained below, but you should know that in the case of a perfectly symmetric object or a symmetrically distributed set of objects, the CoM coincides with the geometric center. And if the mass is unequally distributed, the CoM will be offset toward the locations at which more mass is present.

The CoM is a useful concept in physics when dealing with objects of extended size, because it allows the problem to be simplified -- in particular with problems involving gravity. For example, if you want to calculate the gravitational force from a large extended object like the Sun on another mass at a location outside of the Sun, the Sun's gravity acts as if it were coming from an infinitesimally small object in the center of the Sun but with the same total mass as the Sun. That is, the gravity acts as if it's originating from a point mass (with zero size) with the Sun's mass (just like a on-solar-mass black hole!).

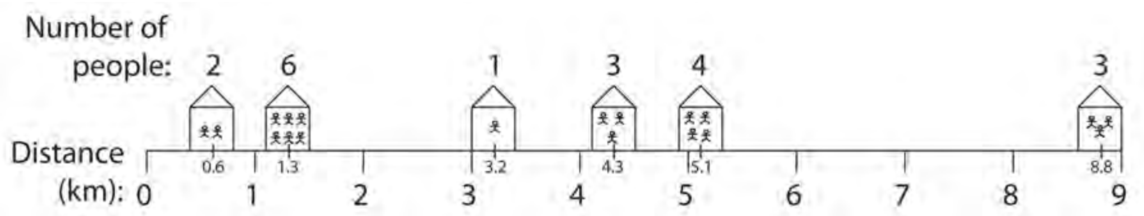
This documents has three sections:

1) Center of Mass for Multiple Objects
2) Center of Mass for a Single Object
3) Center of Related to Torque
4) Additional Resources

We hope you find these explanations helpful, and be sure to visit the referenced websites if you'd like to learn more about this subject.

1. Center of Mass for Multiple Objects

To understand why we say that the center of mass in the Earth-Sun system is actually inside the Sun, a simple analogy might help. Imagine a group of six houses along a road, with a different number of people living in each house. Here's a sketch that shows the number of people in each house and the distance of each house from a reference location defined as "zero kilometers":



Now let's say you want to know the "average location" of the people in this little community, which you might call the "Center of People." Why might you care? Perhaps you're planning to build a store and you want to be as close to as many people as possible.

A nice egalitarian approach to this problem is to multiply the number of people at each location by the distance of that location from zero km (you can think of this as allowing each person to "vote" for their location, so a location with more people will get more votes than a location with fewer people). Here's how that would work out:

2 people at km 0.6 = 1.2 people km
6 people at km 1.3 = 7.8 people km
1 person at km 3.2 = 3.2 people km
3 people at km 4.3 = 12.9 people km
4 people at km 5.1 = 20.4 people km
3 people at km 8.8 = 26.4 people km

Adding up the right column, you get a total of 71.9 people km. If you then divide by the total number of people (19 in this case), you get

71.9 people km/19 people = 3.76 km.

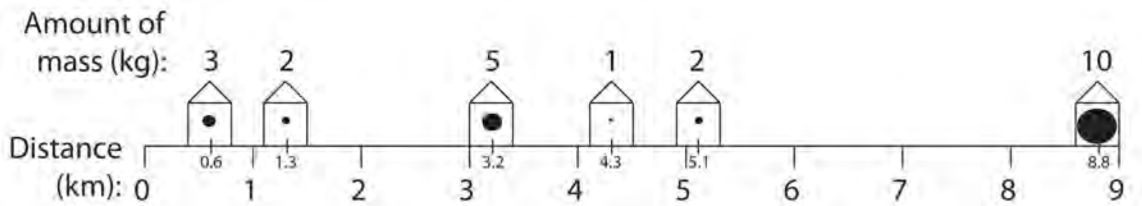
And that's the Center of People in this town. Notice that in this case there are no people at the Center of People – it's just the average location of people (just as the average score on an exam might be 81%, but that doesn't mean that any student got that exact score).

It's easy to write an equation that describes this process. Just call the distance from the reference location "x" and the number of people at each location "P":

$$\text{Center of People distance} = \frac{P_1x_1+P_2x_2+\cdots+P_6x_6}{P_1+P_2+\cdots+P_6} = \frac{\sum P_i x_i}{\sum P_i}$$

where P_1 is the number of people at location x_1, P_2 is the number of people at location x_2 , and so forth.

Now consider a situation in which you have a series of masses instead of people:



In this case, you're not trying to find the Center of People, you're trying to find the center of mass. So instead of multiplying each location by the number of people at that location, multiply by the amount of mass at that location:

3 kg at km 0.6 = 1.8 kg km
2 kg at km 1.3 = 2.6 kg km
5 kg at km 3.2 =16.0 kg km
1 kg at km 4.3 = 4.3 kg km
2 kg at km 5.1 = 10.2 kg km
10 kg at km 8.8=88.0 kg km

Adding up the right column, you get a total of 122.9 kg km. If you then divide by the total mass (23 kg in this case), you get

122.9 kg km/23 kg = 5.34 km.

And that's the center of mass of this little community of masses (and there happens to be no mass at that location in this case).

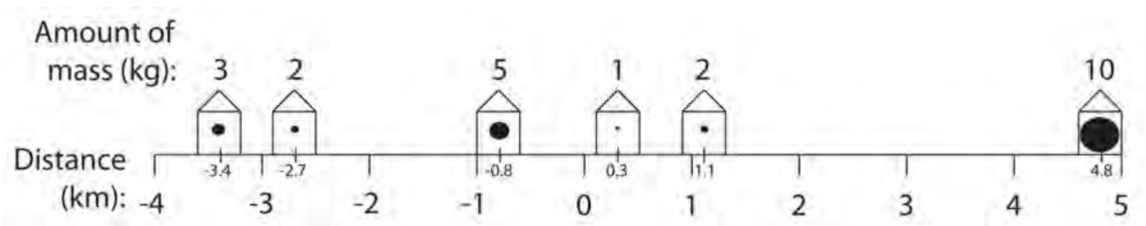
You should also note the center of mass is not the location with half the total mass on one side and half the total mass on the other side – in this case, there are 13 kg to the left of the center of mass and 10 kg to the right of it. That's because you have to consider the location as well as the amount of mass when you're finding the center of mass.

Mathematically, the process looks like this:

$$\text{center of mass distance} = \frac{m_1x_1+m_2x_2+\cdots+m_6x_6}{m_1+m_2+\cdots+m_6} = \frac{\sum m_i x_i}{\sum m_i}$$

where m_1 is the amount of mass at location x_1, m_2 is the amount of mass at location x_2 , and so forth.

It’s important to realize that the location of the zero-km reference point does not change the location of the center of mass. To see that, consider what would happen if you moved your reference point to another location but leave the masses the same:



Now the distances from the reference point have changed, so multiplying each mass by its new distance looks like this:

3 kg at km -3.4 = -10.2 kg km
2 kg at km -2.7 = -5.4 kg km
5 kg at km -0.8 = -4.0 kg km
1 kg at km 0.3 = 0.3 kg km
2 kg at km 1.1 = 2.2 kg km
10 kg at km 4.8= 48.0 kg km

With the new reference point, the right column adds up to 30.9 kg km. Dividing by the total mass (which hasn’t changed) gives 30.9 kg km/23 kg = 1.34 km as the center of mass.

But this is the exact same location, because the reference point (zero km) is now at the location that was formerly called 4 km. And 1.34 km from the new reference point is the same location as 5.34 km from the original reference point.

You may be thinking “That’s fine for one-dimensional problems, where the masses are all lined up. But what about a two- or three-dimensional problem in which the masses are located at different *x, y,* and *z* coordinates?

No problem. The center of mass equation works for each coordinate independently, so you can find the x-value of the center of mass using the x-location of each mass, and you can find the y-value of the center of mass using the y-location of each mass, and you can find the z-value of the center of mass using the z-location of each mass. If the number of masses is N, that looks like this:

Center of Mass x-value = $\frac{m_1x_1+m_2x_2+\cdots+m_Nx_N}{m_1+m_2+\cdots+m_N} = \frac{\sum m_ix_i}{\sum m_i}$

Center of Mass y-value = $\frac{m_1y_1+m_2y_2+\cdots+m_Ny_N}{m_1+m_2+\cdots+m_N} = \frac{\sum m_iy_i}{\sum m_i}$

Center of Mass z-value = $\frac{m_1z_1+m_2z_2+\cdots+m_Nz_N}{m_1+m_2+\cdots+m_N} = \frac{\sum m_iz_i}{\sum m_i}$

How does all this relate to Astronomy and the center of mass of the Earth-Sun system?

Consider the Earth, with mass of about $6x10^{24}$ kg, and the Sun, with mass of $2x10^{30}$ kg, and the distance of about 150 million km between their centers:

To find the center of mass of the Earth-Sun system, just use the same approach described above. Start by multiplying each mass by its location:

$(6x10^{24}$ kg) x $(150x10^{6}$ km) = $9x10^{32}$ kg km
$(2x10^{30}$ kg) x (0 km) = 0 kg km

Adding up the right column gives $9x10^{32}$ kg km, and dividing by the total mass of the Earth and Sun $(6x10^{24}$ kg + $2x10^{30}$ kg) = $2.000006x10^{30}$ kg gives

$(9x10^{32}$ kg km)/$(2.000006x10^{30}$ kg) = 450 km.

That's the distance from the reference point (the center of the Sun in this case) to the center of mass.

Here's how it looks if you use the equation for center of mass for two masses:

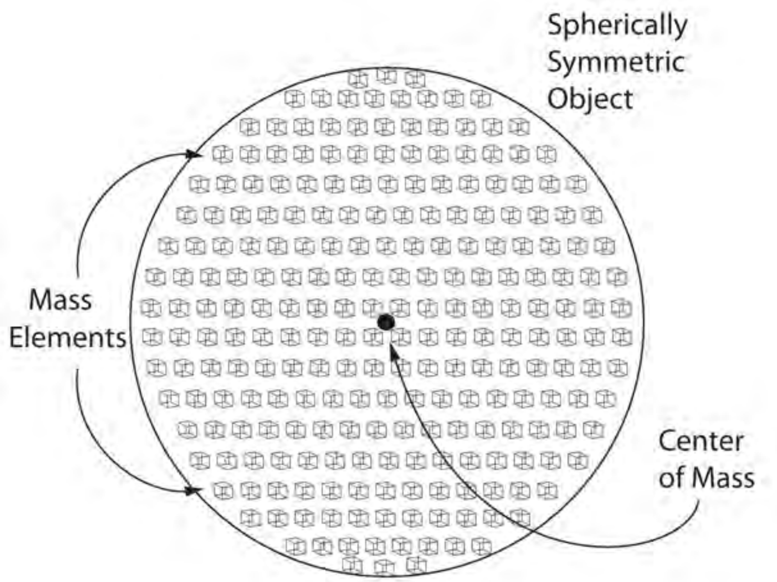
$$\text{Center of Mass distance} = \frac{m_1x_1+m_2x_2}{m_1+m_2} = \frac{(6\times10^{24}kg)(150\times10^{6}km)+(2\times10^{30}kg)(0\ km)}{6\times10^{24}kg+\ 2\times10^{30}kg}$$

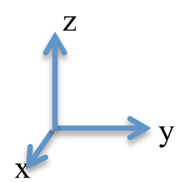
$$= \frac{9\times10^{32}kg\ km}{2{,}000006\times10^{30}kg} = 450\ km$$

2. Center of Mass for a Single Object

The same concept of the "average position of mass" can help you understand why the center of mass concept is useful for a single object in addition to the multiple-object cases described in the previous pages.

The key to understanding the center of mass for a single object is to consider that object (such as a star of a planet) as being composed of multiple small "mass elements" such as those shown in the figure below.





As you can see, we've represented this one object by dozens of discrete mass elements (we could have made them fit the boundaries of the spherical object more closely by making them smaller, but we wanted you to be able to see the individual elements).

We're not saying that stars and planets are made up of tiny cubical elements of mass, we're just asking you to conceptually divide up this continuous spherical object into individual mass elements, each having some amount of the material of the spherical object at its location (that is, each element has its own unique x, y, and z coordinate values).

What's the point of considering these individual bits of mass inside the spherical object? Well, if you followed the discussion on the previous pages about finding the center of mass of a group of objects by multiplying each mass by its position (x, y, or z) and then dividing by the total mass, you can probably guess where the center of mass of a spherically symmetric object is located (and if you can't guess, it's got a black dot over it in the figure).

As you can see, the center of mass of this spherically symmetric object is right at the center of the object (sometimes called the "geometrical center"). You should be able to convince yourself that it must be there by considering the equations for the x, y, and z locations of the center of mass (CoM):

CoM x-value = $\frac{\sum m_i x_i}{\sum m_i}$ CoM y-value = $\frac{\sum m_i y_i}{\sum m_i}$ CoM z-value = $\frac{\sum m_i z_i}{\sum m_i}$

Why do these equations mean the CoM must be at the center of the object? You can understand that by doing a little thought experiment using one of these equations, such as the CoM z-value equation.

Remember that in the CoM equations, z_i represents the distance in the z-direction of each mass m_i from the reference position (and the location of the CoM does not depend on where you put the reference position). But for this thought experiment, it will help to put the reference position at the center of the spherical object.

Now think about how all the mass elements shown in the figure above will determine the z-location of the CoM. When you multiply the mass of an element by its z-position in the numerator of the CoM equation, you can think of that mass element "voting" for its position. But for every mass element at some z-distance above the reference location, there must be an identical element at the same z-distance below the reference location (if there weren't, the object wouldn't be spherically symmetric). And since the z-axis points upward, the element above the reference position will have a positive z-value, and the element below the reference position will have a negative z-value. So when you do the summation called for in the numerator of the CoM equation, the contributions of those two elements will exactly cancel one another.

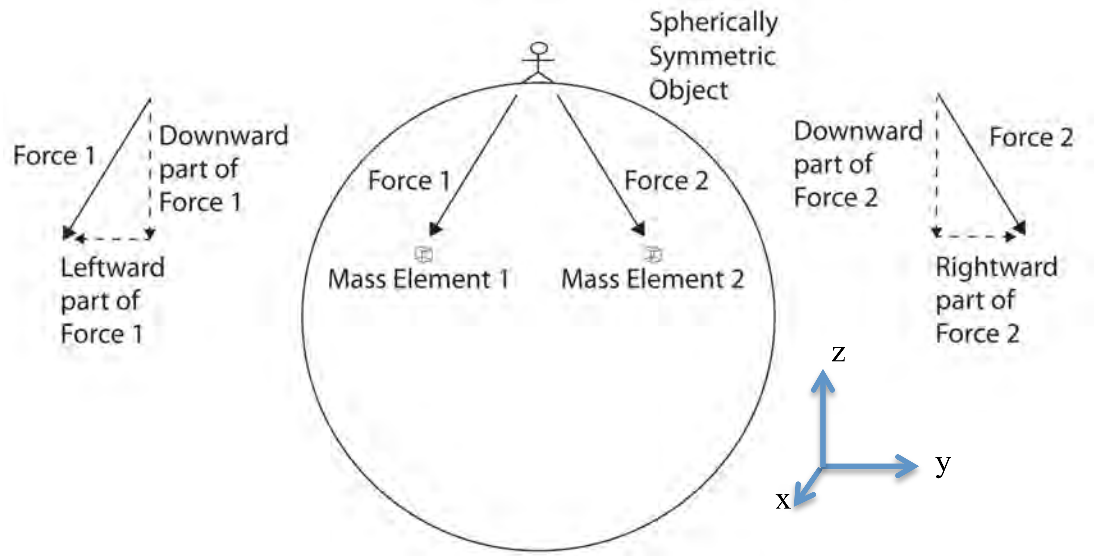
And here's the payoff: since the object is spherically symmetric and every element at positive z that tries to pull the CoM upward must have a corresponding element at negative z trying to pull the CoM downward, and the contributions of those two elements will always cancel. That means that the z-value of the CoM must be zero.

You can make the same argument for mass elements at positive and negative values of x (out of the page and into the page) and for mass elements at positive and negative values of y (right and left of the reference point).

From this argument, you can conclude that the center of mass of a spherically symmetric object must lie at the geometrical center of that object.

Understanding the meaning of the center of mass of a spherically symmetric object can also help you understand why you can treat all the mass of a spherical object as being located at the center when you're using equations such as Newton's Law of Gravity.

To see that, consider the gravitational force that a spherical object produces on a person standing on the surface of that object. Here's a sketch:



The gravitation forces produced by two mass elements are shown in this sketch. These two mass elements are symmetrically located relative to the center of mass (which is the center of the sphere). The mass element to the left of center produces a gravitational force called "Force 1" and the mass element to the right of center produces a gravitational force called "Force 2."

For the person standing on top of the sphere, Force 1 pulls downward (in the negative-z direction) and to the left (in the negative-y direction). Force 2 also pulls downward (in the negative-z direction) but to the right (in the positive-y direction).

So think about the total force on the person due to the gravitational pull of these two elements: one force is down and to the left and the other is down and to the right – but since this object is spherically symmetric, these two mass elements must be pulling equally hard on the person. That means that the leftward part of Force 1 is exactly equal and opposite the rightward part of Force 2 (the downward and leftward parts of Force 1 are illustrated in the side drawing to the left of the sphere, and the downward and rightward parts of Force 2 are illustrated in the side drawing to the right of the sphere – these are called the "components" of the force vectors).

What does this mean? Simply that the left-right (-y and +y) components of the gravitational force vectors for these two mass elements cancel each other, but the downward (-z) components reinforce one another. So the net force produced by these two elements is downward, straight toward the center of mass.

The same argument can be made about any two elements that are symmetrically located in the x-direction or the z-direction, as well. When you add up the forces of all the mass elements within the sphere, the only components that don't cancel are those that point toward the center of mass. So a person standing on the surface will feel the same gravitational force as if all the mass were concentrated exactly at the center of mass.

Using the CoM to calculate the total gravity is far easier than summing up the varying gravity from the front portion of the Sun which is nearer resulting in stronger gravity, and the back portion of the Sun which is farther resulting in weaker gravity, and the rest of the Sun which is intermediate. But these two approaches of

(1) Integrating the cumulative gravity from all the different bits of the Sun at varying distances, and
(2) Pretending all the mass is concentrated at the CoM and calculating the gravity from the total body as if it were a point mass

both give the same answer. So for calculating gravitational force at any point outside an extended object, it is preferable to use the second method for simplicity. That is why, in Chapter 2 on gravity, Section 2.1.1 specifies that the distance between the two objects in Newton's Law of Gravity as the distance between the centers of the two objects -- it is really the distance between their centers of mass. Moreover, in Section 2.1.3 on surface gravity, the relevant distance between person standing on a planet and the planet itself is the distance between their individual centers of mass, which is approximately equal to the radius of the planet.

3. Center of Mass Related to Torque

Center of mass and the related concept of center of gravity is very important in many applications in the fields of Engineering and Architecture. That's because when force is applied directly on an object's center of mass, it experiences no torque (a torque is a force that causes an object to rotate or twist). So if you apply an unbalancd force to an object at the object's CoM, that object will accelerate, but it will not tend turn or tip (or if it's already rotating, the force through the object's CoM will not cause that rotation to speed up or slow down).

This is clearly important for considering the stability of freestanding structures that are subject to forces from gravity or other structures. This relates to astronomical objects, as well; for an object freely floating in outer space, if you could somehow reach inside and push on it exactly at the CoM it would move straight in the direction of that force without rotating (since the torque on the object would be zero). The page below shows animations of how an object would not rotate or rotate, if pushed at the CoM or off the CoM, respectively (animation courtesy of Dr. Dan Russell, Grad. Prog. Acoustics, Penn State):

http://www.acs.psu.edu/drussell/Demos/COM/com-a.html

Granted, it is usually unrealistic to imagine physical reaching inside and poking an object's CoM to exert a force on it, especially if the CoM is deep beneath a solid surface that you can't reach through, or if the object is entirely gaseous with no solid portion to push or pull. However, it is still useful to study this scenario, because this is exactly how the force of gravity works. Gravity is an example of a force that works at a distance, without need of physical contact, and the force of gravity behaves just as if it were physically pulling only on the CoM.

Keep in mind that the CoM is a mathematically-defined point. It can even be a point in empty space; there does not have to be any material actually at that location. In fact when the system in question has multiple objects separated by empty space, as long as the objects have comparable masses and the objects are far apart relative to their physical size, the CoM will typically be somewhere in the empty space between them. But if there is only one object in question, the CoM will typically be somewhere beneath its average surface.

If the single object is bent, curved, or has cavities within it, its CoM can fall outside its physical surface in empty space. That's why we use the qualitative term "average surface". You can imagine this as the imaginary surface that would be created by wrapping an object in a blanket that completely envelops all its protrusions. The CoM will fall somewhere inside the blanket.

For a single object, you can think of the center of mass as the intersection of every possible equilibrium spin axis through the object. That is, imagine inserting a straight rod through the object such that the object would spin around that axis smoothly without any tendency to wobble, or tip the rod. In other words, if you oriented the rod axis horizontally in a gravitational field, the object would have no "heavier side" that would "fall around" to the bottom. Such an axis could be identified for any arbitrary angle through the object, resulting in an infinite number of possible axes. But all of these axes must pass through the CoM. In other words, the CoM is the intersection of all possible equilibrium spin axes through an object. Though technically, you'd only need two such axes to identify the unique CoM, because the two lines would intersect at only one point. The next paragraph offers a practical demonstration of this concept.

The following "PhysicsLAB online" page shows two straightforward ways that you can find the CoM of arbitrary household objects by either balancing them on two fingers (if you are able to lift the object) or using a plumb bob (if you are able to turn the object). These methods are similar to identifying multiple rotation axes, as described in the previous paragraph. A number of links on the page are broken, but the explanations on the page itself are concise and accessible.

http://dev.physicslab.org/Document.aspx?doctype=3&filename=RotaryMotion_CenterMass.xml

For a set of two objects separated by a certain distance, you can visualize their CoM by imagining balancing them on opposite ends of a seesaw. The CoM is the location where you'd have to place the fulcrum to get the seesaw to balance. If the seesaw doesn't balance on your first try, you can adjust the location of the fulcrum to change the distance from it to each object. When the fulcrum coincides with the CoM, the seesaw will balance. (Of course, in a real playground seesaw, you don't move the fulcrum, but rather you can adjust the positions of the masses on the ends, which similarly changes their distances from the pivot point. This allows the riders to adjust the location of their CoM until it aligns with the fulcrum and they become balanced.)

If the objects have identical masses, the balance point -- and thus the fulcrum -- would be exactly halfway between them. (More precisely, it would be exactly halfway between their individual centers of mass.) But if one object is more massive than the other, then the balance point must be closer to the heavier object. You have seen this in action if you've ever tried to balance two unequal-massed people on a seesaw: the larger person must sit closer in toward the fulcrum on her side, and the smaller one farther out toward the end on the opposite side.

The more disparate the masses, the more off-center the CoM will be; it can even be inside the physical surface of the more-massive object. In the limiting case of the mass of the smaller object approaching zero, the overall CoM of the system of masses approaches the individual CoM of the single heavier object. To picture this on the seesaw, imagine trying to balance an adult person with a mosquito: The mosquito's mass is negligible compared to the person, so the person would have to sit essentially directly on top of the fulcrum.

4. Additional Resources:

Western Washington University, Department of Physics and Astronomy

http://faculty.wwu.edu/vawter/PhysicsNet/Topics/Momentum/TheCenterOfMass.html

This page has a good bulleted list of a half-dozen characteristics of the CoM. It also shows some formal mathematical definitions of how to calculate CoM location in 3-D coordinates that is beyond the level of math covered here.

Georgia State University, Department of Physics and Astronomy

http://hyperphysics.phy-astr.gsu.edu/hbase/cm.html

This page shows two ways to set up a mathematical calculation of the CoM for a simple two-body problem in 1-D. The first is for when the CoM is not at the origin (0,0), and the second is for when it is. The first relationship simplifies to the second one when Xcm (the distance of the CoM from the origin) = 0. At the bottom of the page they show increasingly sophisticated mathematics (sums, limits, and integral calculus) for calculating CoM for more complex systems of objects, leading up to a continuous distribution of masses.
A link toward the top of the page takes you to an example problem on a new page that incorporates the ideas of torques, gravitational force, and calculating the CoM for a single extended object:

http://hyperphysics.phy-astr.gsu.edu/hbase/cmms.html

University of Nebraska Lincoln, Astronomy Education group, NAAP - Nebraska Astronomy Applet Project

http://astro.unl.edu/naap/esp/centerofmass.html

This site has a great visualization of CoM between two unequal-mass orbiting bodies, and also an interactive simulation that allows you to change the masses and separation and see how this changes the location of the CoM. They also rearrange the standard simple relationship for CoM of 2 bodies to show how the ratio of the masses is the inverse of the ratio of the distances from the CoM. They give a general relationship that works for more than two bodies as well. Finally, they also perform an example calculation of the CoM of the Jupiter-Sun system.

Ohio State University, Department of Astronomy

http://www.astronomy.ohio-state.edu/~pogge/Ast161/Unit4/orbits.html

Look about halfway down on this page under "Center of Mass". This page points out that the total separation (sum of the distances from the CoM) for 2 bodies is equal to the semi-major axis of the orbit, which relates the idea of CoM to Kepler's Laws. Kepler's laws are summarized further up on this same page, and also covered in more depth in Chapter 2 of our text. This page also has an example calculation for the CoM between the Earth and Sun.