

The Origin of the Equation for Escape Speed

To understand where the escape velocity equation comes from, it helps to consider the energy of an object as it moves away from another object. As you may have learned in a physics class, there are two kinds of energy useful in such situations: kinetic energy and potential energy. Kinetic energy is often called “the energy of motion” because it depends on the object’s speed. Specifically, for an object of mass m traveling at speed v , the object’s kinetic energy (KE) is

$$\text{KE} = \frac{1}{2}mv^2 \quad (1)$$

in which the units of KE are Joules if mass is expressed in kilograms and speed is expressed in meters per second.

The other form of energy relevant to escape speed is gravitational potential energy. Potential energy is often referred to as “the energy of position” because it depends on the distance between objects that are producing forces upon one another. In particular, the change in potential energy of any system is equal to the work done on that system by an outside force. The gravitational potential energy (PE) between two masses (m_1 and m_2) at a distance R from each other is

$$\text{PE} = -G \frac{m_1 m_2}{R} \quad (2)$$

in which G is the universal gravitational constant and the units of the potential energy are Joules if the other quantities are expressed in standard units. The minus sign in this equation reflects the fact that setting the zero point of potential energy is like picking the zero point of a coordinate system, and it’s common to select infinite distance as the zero point of potential energy (which seems reasonable since the force of gravity goes to zero at infinite distance). At any distance less than infinity, the gravitational potential energy between two objects is negative. If this seems counter-intuitive, think of it this way: if two masses are pulling on each other by their mutual gravity, and you (the outside force) want to move one of those masses out to an infinite distance from the other, you would have to do work (that is, you would have to add energy to the system of two masses). And if the potential energy of the system is zero at infinite distance, after you’ve been adding energy by moving the masses apart, then the potential energy must have started off at a value less than zero. So the gravitational potential energy of masses that are close to one another is a big negative number, and that number becomes less negative as you move the masses apart, finally reaching zero at infinite distance.

If the equation for gravitational potential energy reminds you of the equation for the force of gravity between two masses, that’s because the force of gravity is simply the change in gravitational potential energy with distance (and if you’ve had some calculus, you should be able to take the derivative this expression

for the potential energy with respect to R to get the equation for the force of gravity).

How does energy relate to escape speed? It's quite straightforward if you think about the apple thrown vertically upward at a speed exactly equal to v_{esc} . Remember that such an apple reaches zero speed at infinite distance. But zero speed means zero kinetic energy, and infinite distance means zero potential energy. So the total energy (kinetic plus potential) of the system (apple plus Earth) must be zero at that point.

But one of the most important laws of physics is that the energy of a system cannot change unless an outside force acts upon the system. And since no outside force acts upon the apple or Earth once the apple has been launched, the energy of the system must remain the same for the apple's entire journey¹. And if the system's total energy is zero when the apple is at infinite distance, the total energy must be zero at all times. Calling the Earth's mass M and the apple's mass m and setting the total energy (the sum of kinetic plus potential energy) to zero gives

$$\begin{aligned}\text{Total Energy} = \text{KE} + \text{PE} &= \frac{1}{2}mv^2 - G\frac{mM}{R} = 0 \\ \frac{1}{2}mv^2 &= G\frac{mM}{R}\end{aligned}$$

which can be solved for the speed (v):

$$\begin{aligned}\frac{1}{2}mv^2 &= G\frac{mM}{R} \\ v^2 &= 2G\frac{mM}{mR} = 2G\frac{M}{R} \\ v &= \sqrt{\frac{2GM}{R}}\end{aligned}$$

and since this is the speed at which the apple just barely escapes (that is, reaches zero speed at infinite distance), the equation for escape speed is

$$v_{esc} = \sqrt{\frac{2GM}{R}} \quad (3)$$

in which G is the universal gravitational constant, M is the mass of the object from which the other object is escaping, and R is the distance between the centers of the objects.

¹The force of gravity between the Earth and the apple is not an outside force, so it cannot act to change the energy of the system.