Displacement Amplitude Reflection and Transmission Coefficients

This document shows one approach to deriving the displacement amplitude reflection and transmission coefficients (Eqs. 4.27 and 4.28 in *A Student's Guide to Waves*) for transverse mechanical waves on a string.

To understand the displacement amplitude reflection coefficient, begin by considering the force produced on the string by a perfect dashpot matched to the impedance of the string.

As described in the text, a perfect dashpot with matched impedance produces the same drag force on the string as would the remainder of an infinitely long string. At the reflection point, that force is

$$F_{drag} = -Z_1 v_{t,i}$$

in which Z_1 is the impedance of the string and $v_{t,i}$ is the transverse (t) speed of the string produced by the incident (i) wave at the point of reflection[1].

Now consider what happens when the dashpot impedance (call it Z_2) is not matched to the impedance of the string (Z_1). In that case, the drag force produced by the dashpot at the reflection point will be greater or smaller than the force shown above. This "mismatched" drag force is the source of the reflected wave in medium 1, which means it can be written as

$$F_{mismatch} = Z_1 v_{t,r}$$

in which $v_{t,r}$ represents the transverse speed produced by the reflected wave.

At the reflection point, the total force is the sum of the "matching" drag force F_{drag} and the mismatch force $F_{mismatch}$:

$$\begin{aligned} F_{total} &= F_{drag} + F_{mismatch} \\ &= -Z_1 v_{t,i} + Z_1 v_{t,r}. \end{aligned}$$

From the definition of impedance, the total drag force produced by the dashpot (with impedance Z_2) is the negative product of the impedance and the total transverse speed $v_{t,total}$, so

$$-Z_2 v_{t,total} = -Z_1 v_{t,i} + Z_1 v_{t,r}.$$

Since the total transverse speed $v_{t,total}$ is the sum of the transverse speed of the incident wave ($v_{t,i}$) and the transverse speed of the reflected wave ($v_{t,r}$), this means

$$-Z_2(v_{t,i} + v_{t,r}) = -Z_1 v_{t,i} + Z_1 v_{t,r}.$$

Gathering terms involving the incident and reflected transverse speeds gives

$$(Z_1 - Z_2)v_{t,i} = (Z_1 + Z_2)v_{t,r}.$$

[1]Note that this "t" stands for "transverse" and not for "transmitted"

Since the transverse speed is the time derivative of the displacement ψ, this may be written as

$$(Z_1 - Z_2)\left(\frac{\partial\psi}{\partial t}\right)_i = (Z_1 + Z_2)\left(\frac{\partial\psi}{\partial t}\right)_r$$

which may be integrated over time (assuming zero displacement at time $t = 0$), giving

$$(Z_1 - Z_2)\psi_i = (Z_1 + Z_2)\psi_r$$

or

$$\psi_r = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)\psi_i.$$

The displacement amplitude reflection coefficient (r) is defined as the ratio of the displacement amplitudes of the incident and reflected waves, so

$$r = \frac{\psi_r}{\psi_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

which corresponds to Eq. 4.27 in the text.

A similar analysis leads to the displacement amplitude transmission coefficient (t). To see how that works, consider the total displacement amplitude in the portion of the string (with impedance Z_1) to the left of the interface. The total displacement ($\psi_{total,left}$) must be the sum of the displacement amplitude (ψ_i) produced by the incident wave and the displacement amplitude (ψ_r) produced by the reflected wave:

$$\psi_{total,left} = \psi_i + \psi_r.$$

Now consider the displacement amplitude in the portion of the string (with impedance Z_2) in the region to the right of the interface. In that region, the only source of displacement is the transmitted wave, so in that region

$$\psi_{total,right} = \psi_t.$$

But the amplitude of the string displacement must be continuous across the interface (otherwise the string would be broken), so $\psi_{total,left}$ must equal $\psi_{total,right}$, which means

$$\psi_t = \psi_i + \psi_r.$$

Writing ψ_r as $r\psi_i$ makes this

$$\begin{aligned}\psi_t &= \psi_i + r\psi_i \\ &= (1 + r)\psi_i.\end{aligned}$$

The displacement amplitude transmission coefficient (t) is defined as the ratio of the displacement amplitudes of the incident and transmitted waves, so

$$t = \frac{\psi_t}{\psi_i} = 1 + r.$$

Using the expression for the displacement amplitude reflection coefficient r derived above, this becomes

$$\begin{aligned} t = 1 + r &= 1 + \frac{Z_1 - Z_2}{Z_1 + Z_2} \\ &= \frac{Z_1 + Z_2}{Z_1 + Z_2} + \frac{Z_1 - Z_2}{Z_1 + Z_2} \\ &= \frac{2Z_1}{Z_1 + Z_2} \end{aligned}$$

which corresponds to Eq. 4.28 in the text.

For additional information about mechanical waves (as well as waves of other types), we highly recommend Crawford's excellent *Waves* text that is part of the Berkeley Physics Course. Here's the reference:

Crawford, F., *Waves Berkeley Physics Course, Vol. 3*, McGraw-Hill 1965.